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Classical Foundations of Quantum Logic

Claudio Garola[1]

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We construct a language L for a classical first-order predicate calculus with monadic predicates only, extended by means of a family of statistical quantifiers. Then, a formal semantic model $\mathfrak{M}$ is put forward for L which is compatible with a physical interpretation and embodies a truth theory which provides the statistical quantifiers with properties that fit their interpretation; in this framework, the truth mode of physical laws is suitably characterized and a probability-frequency correlation principle is established. By making use of L and $\mathfrak{M}$, a set of basic physical laws is stated that hold both in classical physics (CP) and in quantum physics (QP), which allow the selection of suitable subsets of primitive predicates of L (the set $\mathcal{S}_P$ of pure states; the sets $\mathscr{E}_O$ and $\mathscr{E}_E$ of operational and exact effects, respectively) and the introduction on these subsets of binary relations (a preclusion relation $\#$ on $\mathcal{S}_P$, an order relation $<$ on $\mathscr{E}_E$). By assuming further physical laws, $(\mathscr{E}_E, <)$ turns out to be a complete orthocomplemented lattice [mixtures and atomicity of $(\mathscr{E}_E, <)$ also can be introduced by means of suitable physical assumptions]. Two languages L_E^x and L_E^S are constructed that can be mapped into L; the mapping induces on them mathematical structures, some kind of truth function, an interpretation. The formulas of L_E^x can be interpreted as statements about properties of a physical object, and the truth function on L_E^x is two valued. The formulas of L_E^S can be endowed with two different interpretations as statements about the frequency of some physical property in some class (state) of physical objects; consequently, a two-valued truth function and a multivalued fuzzy-truth function are defined on L_E^S. In all cases the algebras of propositions of these "logics" are complete orthocomplemented lattices isomorphic to $(\mathscr{E}_E, <)$. These results hold both in CP and in QP; further physical assumptions endow the lattice $(\mathscr{E}_E, <)$, hence L_E^x and L_E^S, with further properties, such as distributivity in CP and weak modularity and covering law in QP. In the latter case, L_E^x and L_E^S, together with their interpretations, can be considered different models of the same basic mathematical structure, and can be identified with standard (elementary) quantum logics. These are therefore founded on the classical extended language L with semantic model $\mathfrak{M}$.

[1]Dipartimento di Fisica dell'Università di Lecce, Lecce, Italy.

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INTRODUCTION

According to Jammer (1974),[2] a basic question lies at the origin of quantum logic (QL), that is, whether a new logic which is different from classical logic (CL) is needed in quantum physics (QP), CL being the standard logic underlying the formulation of classical physics (CP).

Beginning with Birkhoff and von Neumann (1936), many authors have given a positive answer to this question and have put forward logical systems that should formalize the underlying logic of QP. Nowadays, after some decades of research in this area, some structures seem to be privileged candidates to the role of models for a quantum sentential calculus (more precisely, for the algebra of propositions of such a calculus), that is, the orthocomplemented lattices which are introduced in many axiomatized or semiaxiomatized approaches to QP from different viewpoints [e.g., the lattice of questions in the Mackey (1963) approach, the lattice of propositions in the Jauch (1968) and Piron (1976) approach, the lattice of events in the Pool (1968) approach, the lattice of decision effects in the Ludwig (1983) approach] which essentially share the same mathematical properties [we have already explored elsewhere, together with other authors, the links between some of the approaches quoted above; see Garola and Solombrino (1983)].

Yet, some authors reject a logical interpretation of these structures, which are interpreted as mere "formalizations of empirical facts," and deny the need for an alternative logic to CL. On the contrary, "a number of theorists proposed regarding quantum logic as a full-fledged new logic which by dictate of experience is due to supersede classical logic" (Jammer, 1974).

Whenever the latter viewpoint is adopted, some further problems arise, even whenever an elementary quantum sentential calculus with basic connectives $\neg$, $\wedge$, $\vee$ only is considered. In fact, the interpretation of the descriptive signs is not necessarily unique [they could represent statements about a single object, or different kinds of statements about ensembles of objects; e.g., Watanabe (1969), Beltrametti and Cassinelli (1976), van Fraassen (1981), Cattaneo and Marino (1988)]; moreover, the interpretation of the connectives is also problematic (e.g., Jauch, 1968; Mielnik, 1976) and, in any case, they should not be read as "not," "and," "or" in the same sense as the corresponding connectives of classical logic (Quine, 1970). More generally, the problem arises of endowing QL with a semantic apparatus which leads to an unambiguous interpretation of the formalism.

[2]Jammer's book contains an extensive bibliography on the subject studied in this paper; further updated bibliography can be found in the book by Beltrametti and Cassinelli (1981) and in Holdsworth and Hooker (1983). Our reference list is by no means complete; the reader is referred to the above texts for a more detailed bibliography.

Many authors have dealt with the latter problem from different viewpoints. For instance, Putnam (1969) and Finkelstein (1972) proposed to "read off" the logic directly from the Hilbert space model for QP (Holdsworth and Hooker, 1983); but this procedure leads to a truth theory for QL which is neither Tarskian nor true-functional. More sophisticated attempts to solve the problem can be found in the treatments usually collected under the name "modal approaches to QL" (Holdsworth and Hooker, 1983). In particular, a modal context allows for some kinds of classical foundation of QL; indeed, Dalla Chiara (1977) has provided a modal interpretation of minimal quantum logic into a suitable modal extension of a classical language. This notwithstanding, it must be noted that any modal approach requires intensional (or semiextensional, Kripke-like) semantics, which increases the complexity (and, according to some authors, the miscomprehensibility) of the interpretative apparatus. More specifically, in modal approaches to QL some concepts are introduced [like van Fraassen's "possible situations" (1981) or Dishkant's "states of knowledge" (1972)] which cannot be identified with the orthodox concept of possible world and rely on an interpretation of physical states which poses some nontrivial epistemological problems. In any case, most approaches in this class seem to accept and corroborate the thesis that a nonstandard logic is required in QP.

Our approach to the problem is completely different. Indeed we intend to defend the following thesis in the present paper: *quantum logical structures can be obtained as particular theories based on specific axioms in the framework of a suitably extended classical language L, with extensional semantics.*

The above thesis is relevant for several reasons. First, according to it, quantum logical axioms do not have a logical status; indeed, they are considered specific axioms of a physical theory. Second, the basic language of the theory is an extended classical language, which suggests that a nonclassical logic is not required in QP (this statement strongly supports the belief of the authors who reject a logical interpretation of quantum logical structures; nevertheless, our results at the end of Section 3 show that a logical interpretation is not completely excluded in our perspective and explain the sense in which it can be adopted, also throwing some light on the interpretation of descriptive and logical signs in QL). Third, the semantic apparatus of L is extensional, hence it dispenses with modality, thus gaining simplicity and avoiding the introduction of possible worlds.

More specifically, our semantic apparatus for L admits a Tarskian truth theory and our interpretation of L matches the intuitive physical interpretation of that part of the common technical language which is formalized by L. It is remarkable that the extensionality of semantics occurs in our approach as a consequence of the fact that physical states are made to correspond to predicates (interpreted inside "laboratories," which are parts of the actual world), not to possible worlds or states of knowledge, thus avoiding a number of epistemological problems. In addition, the extension of any predicate in any laboratory is a set of individual physical systems, or "physical objects," in our semantics, and no physical statement about a physical object is meaningless in our approach (though its truth values could be non-measurable because of quantum mechanical laws).

We add that the independence of the concepts of state and laboratory and the absence in L of any reference to possible worlds makes our semantic model suitable for a Kripkian extension where states, laboratories, and possible worlds are introduced conjointly; this should allow for a more complete characterization of the logical status of physical laws, the characterization in the present paper being incomplete from an epistemological viewpoint, though sufficient for our purposes.

We also note that we introduce a number of concepts while defending our thesis, some of which are epistemologically relevant even independent of the thesis itself. We quote in particular the statistical quantifiers with their semantics, the laboratories, the characterization of the truth mode of deterministic and probabilistic physical laws, and the probability-frequency correlation principle in Section 1; the set of conditions which characterize physical models (in particular, condition SO in Definition 2.1.2), the definitions of fuzzy and exact effects, the treatment of mixtures and entities in Section 2; and the techniques for translating nonstandard languages into a classical language and the characterization of classical physical models in Section 3.

Finally, we underline that some ideas in this paper have been anticipated by other authors. For instance, a distinction between different kinds of quantum logical statements can already be found in van Fraassen (1981); an embedding of the basic poset of physical proposition into a distributive logic has been proposed by Wallace (1979); a measure on the set of possible worlds (here substituted by laboratories) and the definition of conditional probabilities with respect to a given state have been introduced by Bigelow (1979). However, our results stand on a different foundation, and there are basic epistemological and technical differences between our treatment and the ones quoted above.

1. THE LANGUAGE L

In the first part of this section (Sections 1.1–1.6) we introduce a formal language L with a semantic model $\mathfrak{M}$ that we intend to use in order to state physical laws. We want L to be as simple as possible; thus, L is the language of a predicate calculus of the first order with monadic predicates only, extended by means of a new family of quantifiers (the *statistical quantifiers*). Furthermore, $\mathfrak{M}$ is a formal semantic model which is compatible with a suitable intended physical interpretation of L, according to which some predicates in L denote physical states, the elements of a set I of indexes correspond to physical laboratories, and the formulas where statistical quantifiers appear formalize statements about conditional frequencies. We show (Sections 1.7, 1.8) that L can be extended into a language L_Π by means of probability operators, so that probabilistic physical laws can be stated by means of (metalinguistic) schemes of formulas of L_Π, their truth mode being suitably characterized and distinguished from the truth modes of analytical statements or logical laws. Then, we prove that, under some general assumptions (*probability-frequency correlation principle*), a probabilistic statement of L_Π can be substituted by a statement of L about conditional frequencies, so that the extension of L into L_Π can be avoided and physical laws can be stated by means of schemes of formulas of L only.

1.1. Syntax

Following usual procedures, we construct our formal language L by giving an alphabet, i.e., a set of primitive signs classified according to syntactic categories and a (finite) set of formation rules for well-formed formulas. This is done by making use of a nonformalized metalanguage, consisting of a part of the English language together with some technical symbols, i.e., bold letters of the Latin alphabet having the role of metalinguistic variables.

Thus, we introduce the following definitions.

Definition 1.1.1. We call an *alphabet*, and denote by $\mathscr{A}$, the set of the descriptive, logical, and auxiliary signs defined as follows.

Descriptive signs

D1. Individual variables: $x, y, z, \ldots$.

D2. Monadic predicates: $\mathbb{0}, \mathbb{1}$; $E, E_1, \ldots$; $S, S_1, \ldots$.

Logical signs

L1. Connectives: $\neg$, $\wedge$, $\vee$, $\rightarrow$, $\leftrightarrow$.

L2. Quantifiers: $\exists$, $\forall$.

L3. The family of statistical quantifiers $\{\pi_\Delta\}_{\Delta\in\mathscr{B}([0,1])}$, with $\mathscr{B}([0, 1])$ the set of all Borel subsets of the interval $[0, 1]$.

Auxiliary signs

A1. Round parentheses $(\cdot)$; slant $/$.

Furthermore, we denote by X the set of (individual) variables in $\mathscr{A}$ and by $\mathscr{P}$ the set of predicates.

Definition 1.1.2. With reference to Definition 1.1.1, we call *formation rules* (FR) for *well-formed formulas* (*wffs*) the following rules.

W1. For every $\mathbf{x} \in X$ and $\mathbf{P} \in \mathscr{P}$, $\mathbf{P}(\mathbf{x})$ is a wff.

W2. Let **A** be a wff; then, $\neg\mathbf{A}$ is a wff.

W3. Let **A**, **B** be wffs; then $\mathbf{A} \wedge \mathbf{B}$, $\mathbf{A} \vee \mathbf{B}$, $\mathbf{A} \rightarrow \mathbf{B}$, and $\mathbf{A} \leftrightarrow \mathbf{B}$ are wffs.

W4. Let **A** be a wff, **x** an individual variable which occurs free in **A**; then $(\exists\mathbf{x})\mathbf{A}$, $(\forall\mathbf{x})\mathbf{A}$ are wffs.

W5. Let **A**, **B** be wffs, $\mathbf{x} \in X$, **x** free in **A**, and let $\Delta \in \mathscr{B}([0, 1])$; then $(\pi_\Delta \mathbf{x})\mathbf{A}/\mathbf{B}$ is a wff [we shall briefly write $(\pi_\Delta \mathbf{x})\mathbf{A}$ in place of $(\pi_\Delta \mathbf{x})\mathbf{A}/\mathbf{B}$ whenever $\mathbf{B} = \mathbb{1}(\mathbf{x})$].

Furthermore, we denote by Ψ the set of all the wffs constructed by means of the signs in $\mathscr{A}$ and of the rules W_1-W_5, and call *formal language* L the pair $L = (\mathscr{A}, \Psi)$.

1.2. Semantics

A semantic structure for the language L can be introduced by means of the following definitions.

Definition 1.2.1. With reference to the definitions in Section 1.1, we call the *formal statistical semantic interpretation* (briefly, *SS-interpretation*) for the language $L = (\mathscr{A}, \Psi)$ the 5-ple

$$\mathscr{K} = (I, \{D_i, \mathscr{G}_i, \nu_i\}_{i \in I}, \{\mathscr{T}_i\}_{i \in I}, \{\mathscr{W}_i\}_{i \in I}, \rho)$$

defined as follows.

(a) I is a nonempty set, called the set of *laboratories.*

(b) $\{D_i, \mathscr{G}_i, \nu_i\}_{i \in I}$ is a family of measure spaces, which associates to every laboratory $i \in I$ a *domain* D_i, a σ-algebra $\mathscr{G}_i$ of subsets of D_i, and a measure function ν_i defined on all subsets of the algebra $\mathscr{G}_i$ and such that, for every $G \in \mathscr{G}_i$, $\nu_i(G) = 0$ iff G coincides with the empty set $\varnothing$.

(c) $\{\mathscr{T}_i\}_{i \in I}$ is a family which associates with every laboratory $i \in I$ a set $\mathscr{T}_i$ of subsets of D_i that belong to the σ-algebra $\mathscr{G}_i$ and is such that $\varnothing \in \mathscr{T}_i$ and $D_i \in \mathscr{T}_i$.

(d) $\{\mathscr{W}_i\}_{i \in I}$ is a family which associates with every laboratory $i \in I$ a partition $\mathscr{W}_i$ of D_i such that every $W \in \mathscr{W}_i$ belongs to the σ-algebra $\mathscr{G}_i$.

(e) ρ is an assignment function

$$\rho: \quad I \times \mathscr{P} \rightarrow \left(\bigcup_{i \in I} \mathscr{T}_i\right) \cup \left(\bigcup_{i \in I} \mathscr{W}_i\right)$$

such that the following conditions are satisfied.

(i) For every $i \in I$, $\rho(i, \mathbb{0}) = \varnothing$, $\rho(i, \mathbb{1}) = D_i$.

(ii) Two disjoint subsets $\mathscr{E}$, $\mathscr{S}$ of $\mathscr{P}$ exist, with $\mathbb{0}, \mathbb{1} \in \mathscr{E}$, such that $\mathscr{E} \cup \mathscr{S} = \mathscr{P}$, and for every $i \in I$, $\rho(i, \mathscr{E}) = \mathscr{T}_i$, $\rho(i, \mathscr{S}) = \mathscr{W}_i$.

Whenever $\mathcal{H}$ is an SS-interpretation, we call $\mathcal{E}$, $\mathcal{S}$ the sets of symbols of *effects* and *states*, respectively.

In the following definition we collect some derived concepts in any SS-interpretation.

Definition 1.2.2. Let $\mathcal{H}$ be an SS-interpretation for the language $L = (\mathcal{A}, \Psi)$, and let us make reference to the definitions in Section 1.1 and to Definition 1.2.1.

For every $i \in I$ we call the *local assignment function* both the pair (i, ρ) and (by abuse of language) the mapping

$$\rho_i: \quad \mathcal{P} \to \mathcal{T}_i \cup \mathcal{W}_i$$

canonically induced by the pair (i, ρ).

We call the *interpretation of the variables* (of L) every mapping

$$\sigma: \quad I \times X \to \bigcup_{i \in I} D_i$$

such that, for every $i \in I$ and $\mathbf{x} \in X$, $\sigma(i, \mathbf{x}) \in D_i$.

We denote by Σ the set of all interpretations of the variables of L, and for every $\sigma \in \Sigma$ and $\mathbf{x} \in X$ we put

$$\Sigma_{\sigma,x} = \{\sigma_x \in \Sigma \mid \text{for every } i \in I, \mathbf{y} \in X \setminus \{\mathbf{x}\}, \sigma_x(i, \mathbf{y}) = \sigma(i, \mathbf{y})\}$$

For every $\sigma \in \Sigma$ we call the *evaluation* both the pair (ρ, σ) and (by abuse of language) the mapping

$$\rho^\sigma: \quad I \times (X \cup \mathcal{P}) \to \left(\bigcup_{i \in I} D_i\right) \cup \left(\bigcup_{i \in I} \mathcal{T}_i\right) \cup \left(\bigcup_{i \in I} \mathcal{W}_i\right)$$

canonically induced by the pair (ρ, σ).

Finally, for every $i \in I$ and $\sigma \in \Sigma$ we call the *local evaluation* both the triple (i, ρ, σ) and (by abuse of language) the mapping

$$\rho_i^\sigma: \quad X \cup \mathcal{P} \to D_i \cup \mathcal{T}_i \cup \mathcal{W}_i$$

canonically induced by the triple (i, ρ, σ).

1.3. The Intended Interpretation

Our definitions in Sections 1.1 and 1.2 are suggested by the intended physical interpretation of the descriptive signs of L that we have in mind when constructing L.

In order to make this point clear, let us recall that, according to an analysis of physical experiments made by Ludwig (1983; see also Ludwig, 1971, 1977) two sets can be considered fundamental in physics, and precisely the set of *preparation procedures* (here, briefly, *preparations*) and the set of (dichotomic) *registration procedures*. Each of these sets induces on the other, via a probability function, an equivalence relation, so that both can be partitioned into classes that are called *ensembles* (or *states*) and *effects*, and that respectively collect all preparations and all registration procedures which can be considered physically equivalent.

In the present paper we essentially adopt the above analysis (it must be noted, however, that a preparation or registration procedure should not be confused here with its actualization by means of an apparatus in some laboratory; rather, it can be interpreted, extensionally, as the set of all its actualizations in every laboratory, or, intensionally, as a set of rules for constructing the corresponding apparatus in every laboratory). Yet it must be clearly understood that our interpretation of effects and states differs from Ludwig's in some basic aspects; we limit ourselves here to pointing out the specific differences which explain why our mathematical structures do not match the structures introduced by Ludwig in his papers on the subject.

First, the term effect is used here with an enlarged meaning; more precisely, we call effects both the "operational" effects, which can be defined as in Ludwig's approach, and the "conceptual" effects; the latter collect purely theoretical selection procedures and do not correspond to classes of physical apparatus (hence, they play the role of theoretical entities). These different kinds of effects are formally distinguished in Section 3.2, and our theory does not prohibit the set of conceptual effects from being void.

Second, every state, though operationally defined as a class of preparation procedures, as in Ludwig's approach, is represented here by "actual" physical objects in every laboratory; this introduces a conceptual asymmetry between states and effects in our interpretation (discussed in more detail at the end of this subsection) which prohibits the set of preparations from being endowed with the structure type "selection procedure."

Bearing in mind the differences outlined above, let us come to the intended physical interpretation. We state that the predicate signs in $\mathscr{E}$ and $\mathscr{S}$ be interpeted as nouns of effects and states, respectively. Then our interpretation endows the elements of the abstract SS-interpretation introduced in Definition 1.2.1 with a physical meaning, as follows.

(i) Every $i \in I$ is interpreted as a laboratory in the usual physical sense, i.e., a limited space-time domain where measurements and observations take place. It must be clearly understood that every laboratory is conceived here as a part of the actual world and must not be confused with a "possible world" in the sense established by Kripke semantics.

(ii) For every $i \in I$, the domain D_i is interpreted as the set of all the physical objects which are produced in the laboratory i by activating apparatus which realize in i some preparing procedures. Bearing in mind this interpretation, D_i could be assumed to be finite; in this case, the σ-algebra $\mathscr{G}_i$ coincides with the set of all subsets of D_i, and for every $G \in \mathscr{G}_i$ the real number $\nu_i(G)$ may be interpreted as the number of elements in G.

(iii) For every $i \in I$, every class $T \in \mathscr{T}_i$ is the image, through the assignment function ρ_i, of a symbol of effect $\mathbf{E}$. Whenever the effect denoted by

E is "operational" in the sense specified above, T is interpreted as the class of all physical objects in D_i which would give the yes answer if tested with any of the apparatus which actualize in i the registration procedures collected in the effect denoted by **E** (we briefly say in the sequel that T is the class of all physical objects which would be selected in i by the effect denoted by **E**). Yet, since in our intended interpretation conceptual effects are not excluded, some classes of objects may belong to $\mathcal{T}_i$ which would be selected by theoretical procedures, not by means of concrete apparatus in i.

It must be noted that, consistent with the above intended interpretation, we have not assumed any restriction on $\mathcal{T}_i$ (so that two classes T_1, $T_2 \in \mathcal{T}_i$ can be disjoint, or overlap, etc.). Furthermore, we note that the class D_i, which is associated by ρ_i to the effect $\mathbb{1}$, is selected by any apparatus which gives the yes answer whenever a physical object in D_i is tested, while the class $\varnothing$, which is associated by ρ_i to the effect $\mathbb{0}$, is selected by any apparatus which always yields the negative answer.

(iv) For every $i \in I$, every class $W \in \mathcal{W}_i$ is the image, through the local assignment function ρ_i, of a symbol of state **S**. Then, W is interpreted as the class of all physical objects in D_i which are prepared by activating, even iteratively, the apparatus which actualize in i physically equivalent preparation procedures collected in the state denoted by **S** (we briefly say in the sequel that W is the class of all physical objects which are prepared in i according to the state denoted by **S**). Our picture does not take into account dynamical evolution of states after preparation; if such an evolution occurs, we state that any physical object is taken "immediately after" the activation of the preparation procedure.

It must be noted that, consistent with the above intended interpretation, we have assumed that $\mathcal{W}_i$ is a partition of D_i, since the join of all $W \in \mathcal{W}_i$ must be equal to D_i, while two elements W_1, $W_2 \in \mathcal{W}_i$ cannot overlap, since they are prepared by means of physically inequivalent preparation procedures.

(v) For every $i \in I$, every individual variable is associated by the local evaluation ρ_i^σ to a physical object in D_i.

The basic correspondences of our intended interpretation of the descriptive signs of L are thus established. Now, let us explain our above statement about the conceptual asymmetry of states and effects in the present framework. It follows from (iv) that every subset $W \subseteq D_i$ that corresponds to a state in a laboratory i is an *actual* set, since every physical object in it is concretely produced by means of a given preparation at some instant t that belongs to the time interval (which might be infinite) on which i extends [the underlying actuality concept in this definition is classical and goes back to Leibnitz; a different concept has been proposed by other authors, e.g., Prior (1957), which we do not accept here]. Hence, we do not introduce "conceptual" states in our approach. On the contrary, a set T that corresponds to an effect is a *potential* set, since it is conceived of as the set of the physical objects which *would* be selected by means of a suitable registration procedure (should the test be actually done, the objects in T change their state, or also disappear) so that "conceptual" effects are allowed.

1.4. Truth Functions on L.

The family of truth functions on the language L with an SS-interpretation can be introduced by making use of standard techniques, which must be suitably modified and broadened in order to deal with the new symbols in our language, i.e., the statistical quantifiers.

Definition 1.4.1. Let $\mathcal{H}$ be an SS-interpretation for the language $L=(\mathcal{A}, \Psi)$, and let us make reference to the definitions in Sections 1.1 and 1.2.

For every $\mathbf{x}\in X$, $i\in I$, $\sigma\in\Sigma$, we define the mapping

$$\rho_{ix}^{\sigma}: \quad \Psi\to\mathcal{G}_i$$

by means of the following recursive rules.

(i) For every $\mathbf{P}\in\mathcal{P}$ and $\mathbf{y}\in X$,
$\mathbf{x}=\mathbf{y}$ implies $\rho_{ix}^{\sigma}(\mathbf{P}(\mathbf{y}))=\rho_i^{\sigma}(\mathbf{P})$
$\mathbf{x}\neq\mathbf{y}$ and $\rho_i^{\sigma}(\mathbf{y})\in\rho_i^{\sigma}(\mathbf{P})$ imply $\rho_{ix}^{\sigma}(\mathbf{P}(\mathbf{y}))=D_i$
$\mathbf{x}\neq\mathbf{y}$ and $\rho_i^{\sigma}(\mathbf{y})\notin\rho_i^{\sigma}(\mathbf{P})$ imply $\rho_{ix}^{\sigma}(\mathbf{P}(\mathbf{y}))=\varnothing$

(ii) For every $\mathbf{A}\in\Psi$,
$\rho_{ix}^{\sigma}(\neg\mathbf{A})=D_i\backslash\rho_{ix}^{\sigma}(\mathbf{A})$

(iii) For every $\mathbf{A}$, $\mathbf{B}\in\Psi$,
$\rho_{ix}^{\sigma}(\mathbf{A}\wedge\mathbf{B})=\rho_{ix}^{\sigma}(\mathbf{A})\cap\rho_{ix}^{\sigma}(\mathbf{B})$
$\rho_{ix}^{\sigma}(\mathbf{A}\vee\mathbf{B})=\rho_{ix}^{\sigma}(\mathbf{A})\cup\rho_{ix}^{\sigma}(\mathbf{B})$
$\rho_{ix}^{\sigma}(\mathbf{A}\to\mathbf{B})=(D_i\backslash\rho_{ix}^{\sigma}(\mathbf{A}))\cup\rho_{ix}^{\sigma}(\mathbf{B})$
$\rho_{ix}^{\sigma}(\mathbf{A}\leftrightarrow\mathbf{B})=(D_i\backslash(\rho_{ix}^{\sigma}(\mathbf{A})\cup\rho_{ix}^{\sigma}(\mathbf{B})))\cup(\rho_{ix}^{\sigma}(\mathbf{A})\cap\rho_{ix}^{\sigma}(\mathbf{B}))$

(iv) For every $\mathbf{A}\in\Psi$ and $\mathbf{y}\in X$ which occurs free in $\mathbf{A}$,
$\mathbf{x}=\mathbf{y}$ and $\rho_{ix}^{\sigma}(\mathbf{A})=D_i$ imply $\rho_{ix}^{\sigma}((\forall\mathbf{y})\mathbf{A})=D_i$
$\mathbf{x}=\mathbf{y}$ and $\rho_{ix}^{\sigma}(\mathbf{A})\neq D_i$ imply $\rho_{ix}^{\sigma}((\forall\mathbf{y})\mathbf{A})=\varnothing$
$\mathbf{x}\neq\mathbf{y}$ implies $\rho_{ix}^{\sigma}((\forall\mathbf{y})\mathbf{A})=\bigcap_{\sigma_y\in\Sigma_{\sigma,y}}\rho_{ix}^{\sigma_y}(\mathbf{A})$
Analogously,
$\mathbf{x}=\mathbf{y}$ and $\rho_{ix}^{\sigma}(\mathbf{A})\neq\varnothing$ imply $\rho_{ix}^{\sigma}((\exists\mathbf{y})\mathbf{A})=D_i$
$\mathbf{x}=\mathbf{y}$ and $\rho_{ix}^{\sigma}(\mathbf{A})=\varnothing$ imply $\rho_{ix}^{\sigma}((\exists\mathbf{y})\mathbf{A})=\varnothing$
$\mathbf{x}\neq\mathbf{y}$ implies $\rho_{ix}^{\sigma}((\exists\mathbf{y})\mathbf{A})=\bigcap_{\sigma_y\in\Sigma_{\sigma,y}}\rho_{ix}^{\sigma_y}(\mathbf{A})$

(v) For every $\mathbf{A}$, $\mathbf{B}\in\Psi$, $\mathbf{y}\in X$ which occurs free in $\mathbf{A}$, and for every $\Delta\in\mathcal{B}([0,1])$,

$$\mathbf{x}=\mathbf{y} \quad \text{and} \quad \frac{\nu_i(\rho^\sigma_{iy}(\mathbf{A}\wedge\mathbf{B}))}{\nu_i(\rho^\sigma_{iy}(\mathbf{B}))} \in \Delta \quad \text{imply} \quad \rho^\sigma_{ix}((\pi_\Delta \mathbf{y})\mathbf{A}/\mathbf{B}) = D_i$$

$$\mathbf{x}=\mathbf{y} \quad \text{and} \quad \frac{\nu_i(\rho^\sigma_{iy}(\mathbf{A}\wedge\mathbf{B}))}{\nu_i(\rho^\sigma_{iy}(\mathbf{B}))} \notin \Delta \quad \text{imply} \quad \rho^\sigma_{ix}((\pi_\Delta y)\mathbf{A}/\mathbf{B}) = \varnothing$$

$$\mathbf{x}\neq\mathbf{y} \quad \text{implies} \quad \rho^\sigma_{ix}((\pi_\Delta \mathbf{y})\mathbf{A}/\mathbf{B}) = \left\{ \rho_i^{\sigma_x}(\mathbf{x}) \in D_i \,\middle|\, \sigma_x \in \Sigma_{\sigma,x},\ \sigma_x : \frac{\nu_i(\rho^{\sigma_x}_{iy}(\mathbf{A}\wedge\mathbf{B}))}{\nu_i(\rho^{\sigma_x}_{iy}(\mathbf{B}))} \in \Delta \right\}$$

where the convention is made that the quotients are equal to 1 whenever the measure in the denominator is zero [equivalently, whenever the set $\rho^\sigma_{iy}(\mathbf{B})$ or $\rho^{\sigma_x}_{iy}(\mathbf{B})$ is void].

The following proposition collects two basic properties of the mapping ρ^σ_{ix} introduced in Definition 1.4.1.

Proposition 1.4.1. Let $\mathscr{H}$ be an SS-interpretation for the language $L = (\mathscr{A}, \Psi)$, and let us make reference to the definitions in Sections 1.1 and 1.2 and to Definition 1.4.1. Furthermore, let $i \in I$, $\sigma \in \Sigma$, $\mathbf{A} \in \Psi$.

Then, the following statements hold.

(i) For every $\mathbf{x} \in X$, $\rho^\sigma_{ix}(\mathbf{A}) = \rho^{\sigma_x}_{ix}(\mathbf{A})$.

(ii) For every $\mathbf{x} \in X$,

$$\rho_i^\sigma(\mathbf{x}) \in \rho^\sigma_{ix}(\mathbf{A}) \quad \text{iff} \quad \text{for every } \mathbf{y} \in \mathbf{X},\ \rho_i^\sigma(\mathbf{y}) \in \rho^\sigma_{iy}(\mathbf{A})$$

[hence, for every $\mathbf{x}, \mathbf{y} \in X$, $\rho_i^\sigma(\mathbf{x}) \in \rho^\sigma_{ix}(\mathbf{A})$ iff $\rho_i^\sigma(\mathbf{y}) \in \rho^\sigma_{iy}(\mathbf{A})$].

Proof. See Garola (1989).

Definition 1.4.2. Let $\mathscr{H}$ be an SS-interpretation for the language $L = (\mathscr{A}, \Psi)$, and let us make reference to the definitions in Sections 1.1 and 1.2 and to Definition 1.4.1.

Let $\mathscr{P}(I)$ be the power set of I. For every $\sigma \in \Sigma$ we define the mapping

$$\tilde{\rho}^\sigma: \quad \Psi \to \mathscr{P}(I)$$

by means of the following recursive rules.

(i) For every $\mathbf{x} \in X$, $\mathbf{P} \in \mathscr{P}$,

$$\tilde{\rho}^\sigma(\mathbf{P}(\mathbf{x})) = \{ i \in I \mid \rho_i^\sigma(\mathbf{x}) \in \rho_i^\sigma(\mathbf{P}) \}$$

(ii) For every $\mathbf{A} \in \Psi$,

$$\tilde{\rho}^\sigma(\neg\mathbf{A}) = I \backslash \tilde{\rho}^\sigma(\mathbf{A})$$

(iii) For every $\mathbf{A}, \mathbf{B} \in \Psi$,

$$\tilde{\rho}^\sigma(\mathbf{A}\wedge\mathbf{B}) = \tilde{\rho}^\sigma(\mathbf{A}) \cap \tilde{\rho}^\sigma(\mathbf{B})$$
$$\tilde{\rho}^\sigma(\mathbf{A}\vee\mathbf{B}) = \tilde{\rho}^\sigma(\mathbf{A}) \cup \tilde{\rho}^\sigma(\mathbf{B})$$
$$\tilde{\rho}^\sigma(\mathbf{A}\to\mathbf{B}) = (I \backslash \tilde{\rho}^\sigma(\mathbf{A})) \cup \tilde{\rho}^\sigma(\mathbf{B})$$
$$\tilde{\rho}^\sigma(\mathbf{A}\leftrightarrow\mathbf{B}) = (I \backslash (\tilde{\rho}^\sigma(\mathbf{A}) \cup \tilde{\rho}^\sigma(\mathbf{B}))) \cup (\tilde{\rho}^\sigma(\mathbf{A}) \cap \tilde{\rho}^\sigma(\mathbf{B}))$$

(iv) For every $\mathbf{A} \in \Psi$ and $\mathbf{x} \in X$, $\mathbf{x}$ free in $\mathbf{A}$,

$$\tilde{\rho}^{\sigma}((\forall \mathbf{x})\mathbf{A}) = \bigcap_{\sigma_x \in \Sigma\sigma, x} \tilde{\rho}^{\sigma_x}(\mathbf{A})$$

$$\tilde{\rho}^{\sigma}((\exists \mathbf{x})\mathbf{A}) = \bigcup_{\sigma_x \in \Sigma\sigma, x} \tilde{\rho}^{\sigma_x}(\mathbf{A})$$

(v) For every $\mathbf{A}, \mathbf{B} \in \Psi$, $\mathbf{x} \in X$, $\mathbf{x}$ free in $\mathbf{A}$, and $\Delta \in \mathscr{B}([0, 1])$,

$$\tilde{\rho}^{\sigma}((\pi_{\Delta}\mathbf{x})\mathbf{A}/\mathbf{B}) = \left\{ i \in I \,\middle|\, \frac{\nu_i(\rho_{ix}^{\sigma}(\mathbf{A} \wedge \mathbf{B}))}{\nu_i(\rho_{ix}^{\sigma}(\mathbf{B}))} \in \Delta \right\}$$

where the convention is made that the quotient in parentheses is equal to 1 whenever $\rho_{ix}^{\sigma}(\mathbf{B}) = \varnothing$ [or, equivalently, whenever $\nu_i(\rho_{ix}^{\sigma}(\mathbf{B})) = 0$].

The statements in the following proposition exhibit the connection between the mappings ρ_{ix}^{σ} and $\tilde{\rho}^{\sigma}$ introduced in Definitions 1.4.1 and 1.4.2, respectively.

Proposition 1.4.2. Let $\mathscr{H}$ be an SS-interpretation for the language $L = (\mathscr{A}, \Psi)$, and let us make reference to the definitions in Sections 1.1 and 1.2 and to Definitions 1.4.1 and 1.4.2. Furthermore, let $i \in I$, $\sigma \in \Sigma$, $\mathbf{A} \in \Psi$, and $\mathbf{x} \in X$.

Then, the following statements hold.

(i) $\tilde{\rho}^{\sigma}(\mathbf{A}) = \{i \in I \mid \rho_i^{\sigma}(\mathbf{x}) \in \rho_{ix}^{\sigma}(\mathbf{A})\}$
$= \{i \in I \mid \text{for every } \mathbf{y} \in X, \rho_i^{\sigma}(\mathbf{y}) \in \rho_{iy}^{\sigma}(\mathbf{A})\}$

(ii) $\rho_{ix}^{\sigma}(\mathbf{A}) = \{\rho_i^{\sigma_x}(\mathbf{x}) \in D_i \mid \sigma_x \in \Sigma_{\sigma,x}, \sigma_x : i \in \tilde{\rho}^{\sigma_x}(\mathbf{A})\}$

Proof. See Garola (1989).

Definition 1.4.3. Let $\mathscr{H}$ be an SS-interpretation for the language $L = (\mathscr{A}, \Psi)$, and let us make reference to the definitions in Sections 1.1 and 1.2 and to Definition 1.4.2.

For every $i \in I$ and $\sigma \in \Sigma$ we call the *truth function* on Ψ the mapping

$$f_{i\sigma}: \quad \Psi \to \{0, 1\}$$

defined by means of the mapping $\tilde{\rho}^{\sigma}$ as follows:

$$\text{for every} \quad \mathbf{A} \in \Psi, \quad f_{i\sigma}(\mathbf{A}) = 1 \quad \text{iff} \quad i \in \tilde{\rho}^{\sigma}(\mathbf{A})$$

We call a *formal statistical semantic model* for the language $L = (\mathscr{A}, \Psi)$ (briefly, *SS-model*) the pair $\mathfrak{M} = (\mathscr{H}, \{f_{i\sigma}\}_{i \in I, \sigma \in \Sigma})$.

Furthermore, we define the following different modes of truth in the SS-model $\mathfrak{M}$.

(i) Let $i \in I$, $\sigma \in \Sigma$, $\mathbf{A} \in \Psi$; we say that $\mathbf{A}$ is *contingently true* (briefly, $\rho_i^{\sigma} \models \mathbf{A}$) iff $f_{i\sigma}(\mathbf{A}) = 1$.

(ii) Let $i \in I$, $\mathbf{A} \in \Psi$; we say that *i verifies* $\mathbf{A}$ (briefly, $i \models \mathbf{A}$) iff for every $\sigma \in \Sigma$, $f_{i\sigma}(A) = 1$.

(iii) Let $\sigma \in \Sigma$, $\mathbf{A} \in \Psi$; we say that σ *verifies* $\mathbf{A}$ (briefly, $\sigma \models \mathbf{A}$) iff for every $i \in I$, $f_{i\sigma}(\mathbf{A}) = 1$.

(iv) Let $\mathbf{A} \in \Psi$; we say that $\mathbf{A}$ is true in $\mathfrak{M}$ (briefly, $\models_{\mathfrak{M}} \mathbf{A}$) iff for every $i \in I$ and $\sigma \in \Sigma$, $f_{i\sigma}(\mathbf{A}) = 1$.

Finally, for every $\mathbf{A} \in \Psi$ we say that $\mathbf{A}$ is logically true (briefly, $\models_L \mathbf{A}$) iff for every SS-model $\mathfrak{M}$, $\models_{\mathfrak{M}} \mathbf{A}$.

1.5. The Intended Interpretation of L

As observed at the beginning of Section 1.4, the truth function in Definition 1.4.3 is introduced by adapting standard techniques to our present framework. In particular, the mapping $\tilde{\rho}^\sigma$ in Definition 1.4.2, hence $f_{i\sigma}$ in Definition 1.4.3, is defined so that atomic formulas, classical connectives, and classical quantifiers can be endowed with a standard interpretation.

Thus, bearing in mind our intended interpretation in Section 1.3, an atomic wff of the form $\mathbf{E}(\mathbf{x})$, with $\mathbf{x} \in X$ and $\mathbf{E} \in \mathscr{E}$, can be interpreted in terms of natural language as follows:

"The physical object denoted by $\mathbf{x}$ has the property that it would be selected by any procedure, theoretical or not, belonging to the effect denoted by $\mathbf{E}$ (briefly, by the effect denoted by $\mathbf{E}$)."

Furthermore, atomic wffs of the form $\mathbf{S}(\mathbf{x})$, with $\mathbf{x} \in X$ and $\mathbf{S} \in \mathscr{S}$, can be interpreted as follows:

"The physical object denoted by $\mathbf{x}$ has been prepared according to one of the preparations collected in the state denoted by $\mathbf{S}$ (briefly, according to the state denoted by $\mathbf{S}$)."

Finally, every molecular wff where statistical quantifiers do not appear has a standard interpretation.

Let us come now to the formulas in Ψ which contain statistical quantifiers. Here $\tilde{\rho}^\sigma$, hence $f_{i\sigma}$, is defined, by making use of the mapping ρ^σ_{ix}, in such a way that any wff of the form $(\pi_\Delta \mathbf{x})\mathbf{A}/\mathbf{B}$ can be interpreted in terms of natural language about conditional frequencies as follows:

"The physical objects which have the property expressed by $\mathbf{B}$ also have the property expressed by $\mathbf{A}$ with a frequency which lies in the Borel set Δ."

It must be noted that we assume in this interpretation that the measure function ν_i reduces to the number of elements of its argument and that a correspondentistic theory of truth is adopted.

Under the same assumptions, the above interpretation becomes more simple and intuitive whenever $\mathbf{B} = \mathbb{1}(\mathbf{x})$ and $\Delta = \{r\}$, with $r \in [0, 1]$. Indeed, in this case the wff $(\pi_\Delta \mathbf{x})\mathbf{A}/\mathbf{B}$ reduces to $(\pi_r \mathbf{x})\mathbf{A}$, and it can be interpreted as follows:

"The physical objects have the property expressed by **A** with frequency r."

Hence, every molecular wff containing statistical quantifiers can be interpreted by applying recursively classical rules of interpretation for classical connectives and quantifiers together with the above rule. Thus, the interpretation of our formal language L is complete.

It is important to notice that Propositions 1.4.1 and 1.4.2 can be considered as *adequacy* theorems, which establish some properties that are intuitively required to hold, for every $\mathbf{A} \in \Psi$, if the aforesaid conception of truth is adopted. In particular, it follows from statement (i) in Proposition 1.4.2 and from Definition 1.4.3 that $f_{i\sigma}(\mathbf{A}) = 1$ iff for every $\mathbf{x} \in X$, $\rho_i^\sigma(\mathbf{x}) \in \rho_{ix}^\sigma(\mathbf{A})$. This shows that our definitions are consistent with the Tarski conception of truth. We observe explicitly that in the above intuitive interpretation of $(\pi_\Delta \mathbf{x})\mathbf{A}/\mathbf{B}$ no reference to a concept of abstract probability appears. This is an important feature, which we discuss more extensively in the sequel.

Finally, we note that our present semantic model allows the definition of different modes of truth for every $\mathbf{A} \in \Psi$, which are schematized in Definition 1.4.3. This is relevant from an epistemological point of view. Indeed, whenever a wff **A** is such that it is not logically true, but some subset $\Sigma^* \leq \Sigma$ of interpretations of the variables exists such that, for every $\sigma \in \Sigma^*$, σ verifies **A**, then **A** is true in every laboratory under the interpretations in Σ^*, hence it expresses some kind of empirical necessity.

Whenever $\Sigma^* = \Sigma$ this necessity seems to characterize the truth mode of any $\mathbf{A} \in \Psi$ following from a scheme of formulas which formally expresses a physical law (hence any **A** of this kind should be true in $\mathfrak{M}$). In our opinion, the truth mode of such formulas is actually weaker, as we discuss in Section 1.8.

1.6. Statistical Quantifiers

It can be easily recognized that the truth function introduced in Section 1.4 is defined in such a way that all standard properties of the truth function for a classical predicate calculus of the first order hold.

In addition, there are some new properties regarding wffs where statistical quantifiers appear; we limit our attention in this section to these properties only. Then, the following proposition shows that the existential and universal quantifiers can be considered as particular cases of statistical quantifiers, consistent with the intuitive interpretation of the latter discussed in Section 1.5.

Proposition 1.6.1. Let $\mathfrak{M}$ be an SS-model for the language $L = (\mathscr{A}, \Psi)$, and let us make reference to the definition in Sections 1.1 and 1.2 and to

Definition 1.4.3. Furthermore, let $i \in I$, $\sigma \in \Sigma$, $\mathbf{A}$, $\mathbf{B} \in \Psi$, $\mathbf{x} \in X$, with $\mathbf{x}$ free in $\mathbf{A}$.

Then, the following statements hold.

(i) $f_{i\sigma}((\pi_1\mathbf{x})\mathbf{A}/\mathbf{B}) = f_{i\sigma}((\forall\mathbf{x})(\mathbf{B} \to \mathbf{A}))$
[hence, in particular, $f_{i\sigma}((\pi_1\mathbf{x})\mathbf{A}) = f_{i\sigma}((\forall\mathbf{x})\mathbf{A})$].

(ii) $f_{i\sigma}((\pi_0\mathbf{x})\mathbf{A}/\mathbf{B}) = f_{i\sigma}(\neg(\exists\mathbf{x})(\mathbf{B} \to \mathbf{A}))$
[hence, in particular, $f_{i\sigma}(\neg(\pi_0\mathbf{x})\mathbf{A}) = f_{i\sigma}((\exists\mathbf{x})\mathbf{A})$].

Proof. See Garola (1989).

The following proposition collects some semantic properties of the statistical quantifiers which could be easily predicted based on the intuitive interpretation discussed in Section 1.5; this proves that our abstract definition of truth function in Section 1.4 is adequate to the interpretation that we have in mind for our language.

Proposition 1.6.2. Let $\mathfrak{M}$ be an SS-model for the language $L = (\mathscr{A}, \Psi)$, and let us make reference to the definitions in Sections 1.1 and 1.2 and to Definition 1.4.3. Furthermore, let $i \in I$, $\sigma \in \Sigma$, $\mathbf{A}$, $\mathbf{B} \in \Psi$, $\mathbf{x} \in X$.

Then, the following statements hold.

(i) Let $\mathbf{x}$ be free in $\mathbf{A}$. Then, a unique $r \in [0, 1]$ exists such that

$$f_{i\sigma}((\pi_r\mathbf{x})\mathbf{A}/\mathbf{B}) = 1$$

and precisely

$$r = \nu_i(\rho_{ix}^{\sigma}(\mathbf{A} \wedge \mathbf{B}))/\nu_i(\rho_{ix}^{\sigma}(\mathbf{B}))$$

(ii) Let $\mathbf{x}$ be free in $\mathbf{A}$, $r \in [0, 1]$. Then,

$$f_{i\sigma}((\pi_r\mathbf{x})\neg\mathbf{A}) = f_{i\sigma}((\pi_{1-r}\mathbf{x})\mathbf{A})$$

(iii) Let $\mathbf{x}$ be free in $\mathbf{A}$, $\Delta_1, \Delta_2 \in \mathscr{B}([0, 1])$, $\Delta_1 \cap \Delta_2 = \varnothing$. Then,

$$f_{i\sigma}((\pi_{\Delta_1}\mathbf{x})\mathbf{A}/\mathbf{B}) = 1 \quad \text{implies} \quad f_{i\sigma}((\pi_{\Delta_2}\mathbf{x})\mathbf{A}/\mathbf{B}) = 0$$

(iv) Let $\mathbf{x}$ be free in $\mathbf{A}$ and $\mathbf{B}$, $\Delta \in \mathscr{B}([0, 1])$. Then, for every $\sigma_x \in \Sigma_{\sigma,x}$,

$$f_{i\sigma_x}(\mathbf{A}) = f_{i\sigma_x}(\mathbf{B}) \quad \text{implies} \quad f_{i\sigma}((\pi_\Delta\mathbf{x})\mathbf{A}) = f_{i\sigma}((\pi_\Delta\mathbf{x})\mathbf{B})$$

(v) Let $\mathbf{A}$ be closed. Then,

$$i \text{ verifies } \mathbf{A} \quad \text{iff} \quad \mathbf{A} \text{ is contingently true}$$

[equivalently, for every σ_1, $\sigma_2 \in \Sigma$, $f_{i\sigma_1}(\mathbf{A}) = 1$ iff $f_{i\sigma_2}(\mathbf{A}) = 1$].

Proof. See Garola (1989).

1.7. Probability Structure

In the definition of an SS-model for a language L introduced in Section 1.2, the set I of laboratories is not required to be endowed with some algebraic or topological structure. The following definition selects a subclass of models such that I is a probability space.

Definition 1.7.1. Let $\mathfrak{M}$ be an SS-model for the language $L=(\mathscr{A},\Psi)$, and let us make reference to the definitions in Sections 1.1, 1.2, and 1.4.

We say that $\mathfrak{M}$ is endowed with a probability structure (briefly, that $\mathfrak{M}$ is a PSS-model) whenever the following conditions hold.

(i) A σ-algebra Γ of subsets of I is given such that, for every $\sigma\in\Sigma$ and $\mathbf{A}\in\Psi$, $\tilde{\rho}^{\sigma}(\mathbf{A})\in\Gamma$.

(ii) A probability measure μ is defined on Γ [hence the triple (I,Γ,μ) is a probability space in the usual sense].

Let $\mathfrak{M}$ be a PSS-model. Then, the following further levels of truth will be defined, besides the levels introduced in Definition 1.4.3.

Let $\sigma\in\Sigma$, $\mathbf{A}\in\Psi$; we say that σ verifies $\mathbf{A}$ almost everywhere (briefly $\sigma\models^{\text{a.e.}}\mathbf{A}$) iff some $\tilde{I}=^{\text{a.e.}}I$ exists such that, for every $i\in\tilde{I}$, $f_{i\sigma}(\mathbf{A})=1$.

Let $\mathbf{A}\in\Psi$; we say that $\mathbf{A}$ is true almost everywhere in $\mathfrak{M}$ (briefly, $\models^{\text{a.e.}}_{\mathfrak{M}}\mathbf{A}$) iff some $\tilde{I}=^{\text{a.e.}}I$ exists such that, for every $i\in\tilde{I}$, $\sigma\in\Sigma$, $f_{i\sigma}(A)=1$.

1.8. Frequency and Probability

The introduction of a probability structure on $\mathfrak{M}$ in Definition 1.7.1 has been done with the aim of providing a suitable background for characterizing the truth mode of physical laws. But we must still take into account that no sign interpreted in terms of probability has been introduced in L, while probabilistic laws occur in QP; thus, the discussion of the truth mode of physical laws requires an extension of the language L that can be obtained, whenever $\mathfrak{M}$ is a PSS model, by making use of the probability structure in order to formalize the concept of conditional probability that a statement $\mathbf{A}$ be true whenever a statement $\mathbf{B}$ is true. More specifically, one can extend the language $L=(\mathscr{A},\Psi)$ introduced in Definition 1.1.2 into a new language L_Π by means of a family $\{\Pi_\Delta\}_{\Delta\in\mathscr{B}([0,1])}$ of operators which operate on pairs $(\mathbf{A},\mathbf{B})$ of wffs of Ψ, so as to obtain new wffs of the form $\Pi_\Delta(\mathbf{A}/\mathbf{B})$, and by means of a suitable extension to L_Π of the formation rules in Definition 1.1.2; the semantic model $\mathfrak{M}$ can then be extended into a model $\mathfrak{M}_\Pi$ by setting, for every $\sigma\in\Sigma$, $\mathbf{A},\mathbf{B}\in\Psi$, $\Delta\in\mathscr{B}([0,1])$,

$$\tilde{\rho}^{\sigma}(\Pi_\Delta(\mathbf{A}/\mathbf{B}))=I \quad \text{iff} \quad \frac{\mu(\tilde{\rho}^{\sigma}(\mathbf{A}\wedge\mathbf{B}))}{\mu(\tilde{\rho}^{\sigma}(\mathbf{B}))}\in\Delta$$

$$\tilde{\rho}^{\sigma}(\Pi_\Delta(\mathbf{A}/\mathbf{B}))=\varnothing \quad \text{iff} \quad \frac{\mu(\tilde{\rho}^{\sigma}(\mathbf{A}\wedge\mathbf{B}))}{\mu(\tilde{\rho}^{\sigma}(\mathbf{B}))}\notin\Delta$$

and by suitably extending Definitions 1.4.2 and 1.4.3 to the set Ψ_Π of wffs in L_Π. As a consequence of this extension of the semantic model, any wff of the form $\Pi_\Delta(\mathbf{A}/\mathbf{B})$ can be interpreted as follows:

"The value of the conditional probability that the statement $\mathbf{A}$ holds whenever $\mathbf{B}$ holds lies in the Borel set Δ."

We underline that the concepts of frequency and of probability are neatly separated in the extended language L_Π, both at the syntactic and at the semantic level. At the former level, the concept of frequency is formalized by means of *statistical quantifiers*, while the concept of conditional probability for statements is expressed by means of *operators*. At the latter level, the truth function for a statement on frequency is evaluated by counting physical objects inside a laboratory, while the evaluation of the truth function for a statement on probability requires measures on suitable subsets of I. It should also be stressed that the definition of probability introduced here is a statistical definition [in the sense clarified by Carnap (1966)], since it is stated by making reference to a measure on the set of laboratories, each laboratory being a part of the actual world; hence it must not be confused with the "logical" probability that can be introduced, via possible worlds, in Kripke semantics (Hintikka, 1965, 1966).

Now, let us consider a probabilistic physical law; it is apparent that such a law can be formally expressed by means of a (metalinguistic) scheme of formulas of L_Π, and every formula $\mathbf{A}$ of L_Π following from this scheme (which will be said to particularize a probabilistic physical law) takes the form $\mathbf{A} = (\forall \mathbf{x})\Pi_r\mathbf{B}$ [where the convention is adopted of writing Π_r in place of $\Pi_{\{r\}}$, and $\mathbf{B}$ in place of $\mathbf{B}/\mathbb{1}(\mathbf{x})$], with $r \in [0, 1]$, $\mathbf{x}$ an individual variable, $\mathbf{B} \in \Psi_\Pi$, $\mathbf{x}$ free in $\mathbf{B}$, $\mathbf{B}$ closed with respect to every variable $\mathbf{y} \neq \mathbf{x}$ which occurs in it; for simplicity, we assume in the sequel that $\mathbf{B} \in \Psi$. Then, the metalinguistic statement that $\mathbf{A}$ particularizes a physical law is equivalent to assert that, for every $\sigma \in \Sigma$, $\mathbf{B}$ has a probability r in every laboratory (i.e., it is true in a subset of laboratories having measure r); in symbols, $\models_{\mathfrak{M}} \Pi_r\mathbf{B}$ or, equivalently, $\models_{\mathfrak{M}} \mathbf{A}$. Thus, we briefly say that $\models_{\mathfrak{M}}$ denotes the truth mode of a probabilistic physical law in L_Π.

Yet, we do not intend to make use of L_Π and of probability concepts in the present paper. This requires that schemes of formulas of L_Π which express probabilistic laws be substituted by schemes of formulas of L with a suitable truth mode. Such a substitution can be obtained at once in the case of a deterministic physical law, which obviously takes the form $\mathbf{A} = (\forall \mathbf{x})\Pi_1\mathbf{B}$; indeed we get in this case

$$\models_{\mathfrak{M}} \Pi_1\mathbf{B} \quad \text{iff} \quad \text{for every } \sigma\in\Sigma, \quad \tilde{\rho}^\sigma(\Pi_1\mathbf{B}) = \mathrm{I} \quad \text{iff}$$

$$\text{for every } \sigma \in \Sigma, \quad \mu(\tilde{\rho}^\sigma(\mathbf{B})) = 1 \quad \text{iff}$$

$$\text{for every } \sigma \in \Sigma, \quad \tilde{\rho}^\sigma(\mathbf{B}) \overset{\text{a.e.}}{=} I \quad \text{iff}$$

$$\text{for every } \sigma \in \Sigma, \quad \tilde{\rho}^\sigma((\forall \mathbf{x})\mathbf{B}) \overset{\text{a.e.}}{=} I \quad \text{iff} \quad \overset{\text{a.e.}}{\underset{\mathfrak{M}}{\models}} (\forall \mathbf{x})\mathbf{B}$$

Hence, every deterministic physical law can be expressed by means of a scheme of formulas of L whose truth mode is $\models_{\mathfrak{M}}^{\text{a.e.}}$ in L.

In order to generalize this result, we introduce the following probability-frequency correlation principle (briefly, PFC principle).

PFC. Let $\mathfrak{M}$ be a PSS model for the language $L=(\mathscr{A},\Psi)$ and let us make reference to the definitions in Sections 1.1–1.7. Let ε, $\eta\in R^+$, R^+ being the set of positive reals. We say that the measure μ on Γ is ε-reliable within the deviation η if for every $r\in[0,1]$, $\mathbf{x}\in X$, $\mathbf{B}\in\Psi$, $\mathbf{x}$ free in $\mathbf{B}$, $\mathbf{B}$ closed with respect to every variable $\mathbf{y}\neq\mathbf{x}$, the following implication holds:

$$\models_{\mathfrak{M}}\Pi_r\mathbf{B}\quad\text{implies}\quad\text{for every }\sigma\in\Sigma,\quad \mu(\tilde{\rho}^{\sigma}((\pi_{\Delta}\mathbf{x})\mathbf{B}))\geq 1-\varepsilon$$

with $\Delta=[r-\eta, r+\eta]\cap[0,1]$.

Then, for every ε, $\eta\in R^+$, every SS-model $\mathfrak{M}$ which bears the physical interpretation specified in Section 1.3 can be endowed with a probability structure such that the measure μ is ε-reliable within the deviation η.

Our introduction of the PFC principle can be intuitively justified as follows. Whenever a statement **A** is made in physics about the probability that a physical system has the property expressed by another statement **B**, there are some pretheoretical methodological criteria which allow us to select in I a subset $\tilde{I}$ of laboratories, such that we can "*a priori*" predict that the frequency of the aforesaid property approaches probability within a prefixed interval in a prefixed percentage of laboratories in $\tilde{I}$ (in particular, the "law of large numbers" of statistics can be assumed as basic criterion; in this case, $\tilde{I}$ is such that for every $i\in\tilde{I}$ the domain D_i contains a "large" number of physical objects). Then, the measure μ can be concentrated on $\tilde{I}$, so that the PFC principle holds.

Now, let us assume that $\mathfrak{M}$ is a PSS model with a measure μ which is ε-reliable within the deviation η, with ε, $\eta\in R^+$. Whenever ε, η are "small" we can introduce the following approximations:

(i) $\Delta=\{r\}$.

(ii) $\mu(\tilde{\rho}^{\sigma}((\pi_r\mathbf{x})\mathbf{B}))=1$.

Whenever (i) and (ii) hold, the implication in the PFC principle simplifies as follows:

$$\models_{\mathfrak{M}}\Pi_r\mathbf{B}\quad\text{implies}\quad\text{for every }\sigma\in\Sigma,\quad \tilde{\rho}^{\sigma}((\pi_r\mathbf{x})\mathbf{B})\overset{\text{a.e.}}{=} I$$

that is,

$$\models_{\mathfrak{M}}\Pi_r\mathbf{B}\quad\text{implies}\quad\models_{\mathfrak{M}}^{\text{a.e.}}(\pi_r\mathbf{x})\mathbf{B}$$

The above implication is sufficient to conclude that the probabilistic physical law $\Pi_r \mathbf{B}$ can be substituted by the scheme $(\pi_r \mathbf{x})\mathbf{B}$ of formulas of L, whose particularizations are required to be true almost everywhere in $\mathfrak{M}$ according to Definition 1.7.1 (the substitution implies that μ be suitably chosen and holds with the degree of approximation that has been specified above). Indeed, the metalinguistic statement $\models_{\mathfrak{M}} \Pi_r \mathbf{B}$ must be considered a theoretical statement in physics (the set of laboratories is not necessarily finite, while a test of the truth value of $\Pi_r \mathbf{B}$ would require a test of $\mathbf{B}$ in every laboratory), the empirical implications of which are exhausted by the statement $\models_{\mathfrak{M}}^{\text{a.e.}} (\pi_r \mathbf{x})\mathbf{B}$, if the measure μ is suitably chosen, within the limits of our above approximation.

We often make use of the aforesaid substitution in the sequel, when we state physical laws in terms of frequency (since we make use of schemes of formulas of L only) instead of in terms of probability. Whenever this procedure occurs, we say that we operate "in statistical approximation."

2. PSS MODELS FOR PHYSICAL LANGUAGES

The main purpose of the present section (Sections 2.1–2.7) is to translate into our present framework, via the intended physical interpretation, basic semantic relations that hold in the common physical language of CP and QP; some are conventions that make our language easier to handle, some are known physical laws, and some are physical assumptions that are usually left implicit, though they have the truth mode of physical laws. Here they take the form of metalinguistic conditions on the semantic model $\mathfrak{M}$, and can be reduced in most cases to schemes of formulas of L with a suitable truth mode.

By making use of the above conditions, a number of results can be obtained. In particular, we introduce a new characterization of pure states, a preclusivity relation $\#$ on the set of pure states, and an (empirical) partial order $<$ on the set of the effects; then, we suitably characterize the subsets $\mathscr{E}_O$ and $\mathscr{E}_E$ of operational (or fuzzy) and exact effects, respectively (the latter corresponding to exact procedures which test physical properties) and show that the poset $(\mathscr{E}_E, <)$ turns out to be a complete orthocomplemented lattice, which is a well-known result in many axiomatic approaches to QP. Finally, in Sections 2.6–2.8 the atomicity of $(\mathscr{E}_E, <)$, nonpure states, and entities are discussed.

2.1. Specialized PSS Models

We select the subclass of the irredundant PSS-models by means of two conditions which formalize pretheoretical requirements on our language that are discussed in Section 2.2.

Definition 2.1.1. Let $\mathfrak{M}$ be a PSS-model for the language $L=(\mathscr{A}, \Psi)$, and let us make reference to the definitions in Section 1.

We say that $\mathfrak{M}$ satisfies the condition of *bijectivity on state symbols* (briefly, SB condition) whenever the following statement holds:

SB. For every $i \in I$, the local assignment function ρ_i maps bijectively $\mathscr{S} \backslash \text{Kern } \rho_i$ onto $\mathscr{W}_i$.

We say that $\mathfrak{M}$ satisfies the condition of *realizability of predicates* (briefly, PR condition) whenever the following statement holds:

PR. For every finite subset $\mathscr{Q} \subseteq \mathscr{P}$, some subset $I_{\mathscr{Q}} \subseteq I$ exists, with $\mu(I_{\mathscr{Q}}) \neq 0$, such that:

(i) For every $i \in I_{\mathscr{Q}}$ and $P \in \mathscr{Q}$, $\rho_i(P) \neq \varnothing$.

(ii) For every $\mathbf{E}_1, \mathbf{E}_2 \in \mathscr{E} \cap \mathscr{Q}$,

$$\mathbf{E}_1 \neq \mathbf{E}_2 \quad \text{implies} \quad \text{for every } i \in I_{\mathscr{Q}}, \quad \rho_i(\mathbf{E}_1) \neq \rho_i(\mathbf{E}_2)$$

Finally, we say that $\mathfrak{M}$ is *irredundant* whenever it satisfies the SB and the PR conditions.

In the subclass of irredundant PSS models a more specialized subclass can be selected by means of further conditions that formalize physical properties which hold both in classical and in quantum physics, as discussed in more detail in Section 2.2.

Definition 2.1.2. Let $\mathfrak{M}$ be an irredundant PSS model for the language $L=(\mathscr{A}, \Psi)$, and let us make reference to the definitions in Section 1 and to Definition 2.1.1.

We say that $\mathfrak{M}$ satisfies the condition of *observability of states* (briefly, SO condition) whenever the following statement holds.

SO. A mapping $g: \mathscr{S} \to \mathscr{E}$ and a subset $\tilde{I} \overset{\text{a.e.}}{=} I$ exist such that:

(i) For every $i \in \tilde{I}$, $\mathbf{S} \in \mathscr{S}$, $\rho_i(\mathbf{S}) \subseteq \rho_i(g(\mathbf{S}))$.

(ii) For every $i \in \tilde{I}$, $\mathbf{S} \in \mathscr{S}$, $T \in \mathscr{T}_i$,

$$\rho_i(\mathbf{S}) \subseteq T \quad \text{implies} \quad \rho_i(g(\mathbf{S})) \subseteq T$$

Let $\mathfrak{M}$ satisfy the SO condition. We denote by $\#$ the binary relation on $\mathscr{S}$ defined as follows.

For every $\mathbf{S}_1, \mathbf{S}_2 \in \mathscr{S}$,

$$\mathbf{S}_1 \# \mathbf{S}_2 \quad \text{iff} \quad \text{for every } i \in \tilde{I}, \rho_i(\mathbf{S}_1) \cap \rho_i(g(\mathbf{S}_2)) = \varnothing.$$

Furthermore, for every $\mathbf{S} \in \mathscr{S}$ we say that $\mathbf{S}$ is a symbol of a pure state if it belongs to the bijectivity subdomain of g, which is denoted by $\mathscr{S}_P$; we still denote by $\#$ (by abuse of language) the restriction to $\mathscr{S}_p$ of the relation $\#$.

Finally, we say that $\mathfrak{M}$ satisfies the condition of *pure states symmetry* (briefly, SY condition) iff the following statement holds.

SY. For every $i \in \tilde{I}$ and $\mathbf{S}_1, \mathbf{S}_2 \in \mathscr{S}_P$,

$$\frac{\nu_i(\rho_i(g(\mathbf{S}_2)) \cap \rho_i(\mathbf{S}_1))}{\nu_i(\rho_i(\mathbf{S}_1))} = \frac{\nu_i(\rho_i(g(\mathbf{S}_1)) \cap \rho_i(\mathbf{S}_2))}{\nu_i(\rho_i(\mathbf{S}_2))}$$

The following propositions state some basic consequences of the SO and SY conditions.

Proposition 2.1.1. Let $\mathfrak{M}$ be an irredundant PSS model for the language $L = (\mathscr{A}, \Psi)$, and let us make reference to the definitions in Section 1 and to Definitions 2.1.1 and 2.1.2.

Let $\mathfrak{M}$ satisfy the SO and SY conditions. Then, the following statements hold.

(i) For every $\mathbf{S}_1$, $\mathbf{S}_2 \in \mathscr{S}_P$,

$$\mathbf{S}_1 = \mathbf{S}_2 \quad \text{iff} \quad \text{for every } i \in \tilde{I}, \quad \rho_i(\mathbf{S}_2) \subseteq \rho_i(g(\mathbf{S}_1)).$$

(ii) the binary relation $\#$ is nonreflexive and symmetric [briefly, $\#$ is a preclusivity relation (Cattaneo and Marino, 1988)] on $\mathscr{S}_P$.

Proof. See Garola (1989).

By making use of the definitions in Section 1 and of Definitions 2.1.1 and 2.1.2, conditions SB, PR, SO, and SY can be restated as metalinguistic schemes of formulas of L whose truth mode is evidenced by means of the symbols introduced in Sections 1.4.3 and 1.7.1 (Garola, 1989).

2.2. The Intended Interpretation for Specialized PSS Models

We intend to discuss in this subsection the interpretation of the definitions and propositions stated in Section 2.1.

Let us begin with the SB condition. Bearing in mind our intended interpretation, let us require that different symbols of state denote different classes of physically equivalent preparation procedures; then, in every laboratory, they must be mapped by the assignment function into different (actually, disjoint) sets of physical objects whenever they are not mapped into the empty set. It is apparent that our condition SB formalizes this economy condition on symbols.

Second, let us consider condition PR. It is trivially true in every physical theory that the elements of every finite subset of effects or states can be simultaneously represented by means of nonempty subsets of physical objects, and that this can be done in a set of laboratories which is as large as desired. Then, (i) in condition PR formalizes this property.

Furthermore, let us require that different symbols of effects correspond to different classes of physically equivalent registration procedures; then they must be mapped by the assignment function on sets of physical objects which do not coincide (though they may overlap) in a number of laboratories which is as large as desired. Statement (ii) in condition PR obviously formalizes this requirement.

It should be noted that the SB and PR (metalinguistic) conditions have a different logical status with respect to the SO and SY conditions in Definition 2.1.2. Indeed, the former can be fulfilled by suitably modifying (if needed) the set of predicates, while the latter impose that some relations on sets of physical objects inside (almost) every laboratory hold, as we shall presently discuss. Thus, the SB and PR conditions establish analytical properties of the language *L*, while the SO and SY conditions impose that some physical laws hold which can be expressed by means of statements of *L*.

Let us comment now on Definition 2.1.2, and let us begin with condition SO. By analyzing CP and QP, one concludes that in both theories the following property can be proved to hold. Let us consider any physical state which is actualized in a laboratory i by a set of apparatus that prepares a class W of physical objects; then, some effect can be made to correspond to this state, which is actualized in the laboratory i by a set of apparatus that select a class T of physical objects, such that (i) it is "almost certain" for the given state (i.e., in almost every laboratory i any object in W gives the yes answer if tested with an apparatus in the effect, so that $W \subseteq T$); (ii) it is minimal in the class of all effects which are almost certain for the given state (i.e., in almost every laboratory i the class T is minimal in the set of the classes selected by apparatus which give the yes answer for every object in W). Indeed, this effect can be considered as the effect which tests "whether a physical object is in the corresponding state." Then, it follows from the intended interpretation discussed in Section 1.3 that the assumption about the existence of a mapping g and statements (i), (ii) in condition SO formalize the above property.

It should be stressed that this property holds both in CP and in QP. It should also be noted that in both physical theories the correspondence between states and minimal effects is not one-to-one, so that the pure states can be characterized by the property that different pure states are associated with different minimal effects. This justifies, via the intended interpretation, the name "symbols of pure states" given to the elements of the bijectivity subdomain of g in Definition 2.1.2.

It is relevant to observe that, consistent with our above physical interpretation, we expect that for every pair of symbols of pure states $\mathbf{S}_1$, $\mathbf{S}_2$, with $\mathbf{S}_1 \neq \mathbf{S}_2$, and for every $i \in \tilde{I}$, the set $\rho_i(\mathbf{S}_2) \cap \rho_i(g(\mathbf{S}_1))$ of the physical objects prepared according to the state denoted by $\mathbf{S}_2$ which are selected by the minimal effect denoted by $g(\mathbf{S}_1)$ and associated with $\mathbf{S}_1$ must be assumed to be void in CP, while it can be nonvoid in QP if the states denoted by $\mathbf{S}_1$ and $\mathbf{S}_2$ are suitably chosen; in any case the inclusion $\rho_i(\mathbf{S}_2) \subseteq \rho_i(g(\mathbf{S}_1))$ cannot be true almost everywhere, because of Proposition 2.1.1, statement (i). We believe that this remark focuses one of the deeper differences between CP and QP; in fact, the quantum behavior illustrated above establishes the breakdown of strict determinism in the latter theory. Indeed, different physical objects in $\rho_i(\mathbf{S}_2)$, though equivalently prepared (i.e., satisfying identical boundary conditions), since they belong to the realization in i of the same pure state, may exhibit different physical properties in QP, some of them sharing the property corresponding to $g(\mathbf{S}_1)$, some of them not.

Let us come to the relation $\#$. The basic idea for introducing it is that two pure states can be considered "completely different" whenever in almost every laboratory no physical object prepared according to one of them is selected by the minimal effect associated to the other one; this is exactly what occurs, according to CP, in almost every laboratory iff the states are different, and according to QP iff they are "orthogonal" (that is, the projection operator that represents one of them is orthogonal to the projection operator that represents the other). Thus, $\#$ can be identified with known relations in CP and in QP whenever our intended interpretation in Section 1.3 is adopted.

Let us discuss now the SY condition. According to our intended interpretation, this condition says that the selection frequency of physical objects prepared according to a given state by means of apparatus which test whether these objects are in another state is equal to the selection frequency that occurs whenever the roles of the two states are exchanged. Therefore, the SY condition establishes a kind of symmetry between pure states which trivially holds in CP. In QP a similar symmetry holds; yet it is expressed in terms of probabilities, not in terms of frequencies. According to our analysis in Section 1.8 of probabilistic physical laws, every statement about probability can be substituted by a statement about frequencies "in statistical approximation" (the meaning of this expression has been specified in detail in Section 1.8), inside every laboratory, where only frequencies can be actually evaluated (the intuitive basis for the substitution is the following: we expect that the differences between frequencies and probability values are small in most laboratories whenever the number of physical objects that are taken into account is high). Therefore, the SY condition, which holds exactly in CP, must be considered in QP as formalizing the above quantum symmetry between states in the statistical approximation.

Finally, we note that, whenever the SY condition holds, the binary relation # is a preclusivity relation on $\mathcal{S}_P$ [Proposition 2.1.1, (ii)], hence, via the intended interpretation, on pure states; this result is consistent with our above remarks on the particularization of # in CP and in QP, and it gives further support to our decision of stating the SY condition as a basic condition for our semantic model.

2.3. Mathematical Interlude

In the following proposition we collect some results that will be used throughout in the sequel.

Proposition 2.3.1. Let $\mathcal{S}$ be a set, let # be a preclusivity (i.e., non-reflexive and symmetric) relation on $\mathcal{S}$, and let $\mathcal{P}(\mathcal{S})$ be the power set of $\mathcal{S}$. Then, the mapping

$$^{\perp}: \quad H \in \mathcal{P}(\mathcal{S}) \to H^{\perp} = \{S \in \mathcal{S} \mid \text{for every } \mathbf{S}^* \in H, \mathbf{S} \# \mathbf{S}^*\} \in \mathcal{P}(\mathcal{S})$$

is a weak orthocomplementation on $(\mathcal{P}(\mathcal{S}), \subseteq)$.[3]

Furthermore, let us put $\mathcal{L} = \{H \in \mathcal{P}(\mathcal{S}) \mid H = H^{\perp\perp}\}$. Then, the following statements hold.

(i) $\varnothing, \mathcal{S} \in \mathcal{L}$, and $\varnothing^{\perp} = \mathcal{S}$, $\mathcal{S}^{\perp} = \varnothing$.

(ii) $(\mathcal{L}, \subseteq)$ is a complete lattice, with minimal element $\varnothing$ and maximal element $\mathcal{S}$.

(iii) Let $\sqcap$ and $\sqcup$ denote meet and joint in $(\mathcal{L}, \subseteq)$ respectively. Then, for every subset $\mathcal{V} \in \mathcal{L}$,

$$\sqcap_{H \in \mathcal{V}} H = \bigcap_{H \in \mathcal{V}} H$$

$$\sqcup_{H \in \mathcal{V}} H = \left(\sqcap_{H \in \mathcal{V}} H^{\perp} \right)^{\perp}$$

(iv) The restriction $^{\perp}$ (by abuse of language) of the mapping $^{\perp}$ to $\mathcal{L}$ is a standard orthocomplementation[4] on $(\mathcal{L}, \subseteq)$.

Proof. Straightforward. ■

2.4. Fuzzy Effects and the Lattice of Exact Effects

The following definition introduces some mappings and relations which restate in our present framework known concepts in the foundations of QP.

[3]Equivalently, the following statements hold. (i) For every $H \in \mathcal{P}(\mathcal{S})$, $H \subseteq H^{\perp\perp}$. (ii) For every $H \in \mathcal{P}(\mathcal{S})$, $H \cap H^{\perp} = \varnothing$. (iii) For every $H_1, H_2 \in \mathcal{P}(\mathcal{S})$, $H_1 \subseteq H_2$ implies $H_2^{\perp} \subseteq H_1^{\perp}$.

[4]Equivalently, the following statements hold. (i) For every $H \in \mathcal{L}$, $H = H^{\perp\perp}$. (ii) For every $H \in \mathcal{L}$, $H \sqcap H^{\perp} = \varnothing$ and $H \sqcup H^{\perp} = \mathcal{S}$. (iii) For every $H_1, H_2 \in \mathcal{L}$, $H_1 \subseteq H_2$ implies $H_2^{\perp} \subseteq H_1^{\perp}$.

Definition 2.4.1. Let $\mathfrak{M}$ be an irredundant PSS model for the language $L = (\mathscr{A}, \Psi)$, and let us make reference to the definitions in Sections 1 and 2.1. Furthermore, let $\mathfrak{M}$ satisfy the SO and SY conditions.

We denote by S_T and S_F the mappings of $\mathscr{E}$ into the power set $\mathscr{P}(\mathscr{S}_P)$ defined as follows:

$$S_T: \quad \mathbf{E} \in \mathscr{E} \to \{\mathbf{S} \in \mathscr{S}_P \mid \text{for every } i \in \tilde{I}, \rho_i(\mathbf{S}) \subseteq \rho_i(\mathbf{E})\} \in \mathscr{P}(\mathscr{S}_P)$$

$$S_F: \quad \mathbf{E} \in \mathscr{E} \to \{\mathbf{S} \in \mathscr{S}_P \mid \text{for every } i \in \tilde{I}, \rho_i(\mathbf{S}) \cap \rho_i(\mathbf{E}) = \varnothing\} \in \mathscr{P}(\mathscr{S}_P)$$

and for every $\mathbf{E} \in \mathscr{E}$ we call $S_T(\mathbf{E})$ the *certainly yes domain* of $\mathbf{E}$, and $S_F(\mathbf{E})$ the *certainly no domain* of $\mathbf{E}$.

We denote by $<$ the quasi-order relation on $\mathscr{E}$ defined as follows:

$$\text{for every } \mathbf{E}_1, \mathbf{E}_2 \in \mathscr{E}, \quad \mathbf{E}_1 < \mathbf{E}_2 \quad \text{iff} \quad S_T(\mathbf{E}_1) \subseteq S_T(\mathbf{E}_2)$$

and denote by $\approx$ the equivalence relation on $\mathscr{E}$ defined as follows:

$$\text{for every } \mathbf{E}_1, \mathbf{E}_2 \in \mathscr{E}, \quad \mathbf{E}_1 \approx \mathbf{E}_2 \quad \text{iff} \quad \mathbf{E}_1 < \mathbf{E}_2 \text{ and } \mathbf{E}_2 < \mathbf{E}_1$$

[equivalently, $\mathbf{E}_1 \approx \mathbf{E}_2$ iff $S_T(\mathbf{E}_1) = S_T(\mathbf{E}_2)$].

Furthermore, the following definition shows that complete orthocomplemented lattices can appear in physics because of the existence of a preclusivity relation on the set of pure states.

Definition 2.4.2. Let $\mathfrak{M}$ be an irredundant PSS model for the language $L = (\mathscr{A}, \Psi)$, and let us make reference to the definitions in Sections 1 and 2.1, and to Proposition 2.3.1. Furthermore, let $\mathfrak{M}$ satisfy the SO and SY conditions, and let $\mathscr{P}(\mathscr{S}_P)$ be the power set of $\mathscr{S}_P$.

We denote by $^\perp$ the weak orthocomplementation on $(\mathscr{P}(\mathscr{S}_P), \subseteq)$ defined as follows:

$$^\perp: \quad H \in \mathscr{P}(\mathscr{S}_P) \to H^\perp = \{\mathbf{S} \in \mathscr{S}_P \mid \text{for every } \mathbf{S}^* \in H, \mathbf{S} \# \mathbf{S}^*\} \in \mathscr{P}\text{-}\mathscr{S}_P)$$

Moreover, we put

$$\mathscr{L} = \{H \in \mathscr{P}(\mathscr{S}_P) \mid H = H^{\perp\perp}\}$$

and denote by $\bigcup$ and $\bigcap$ join and meet, respectively, in the complete orthocomplemented lattice $(\mathscr{L}, \varnothing, \subseteq, ^\perp)$ with minimal element $\varnothing$.

We can now formally introduce the subsets of operational and exact effects.

Definition 2.4.3. Let $\mathfrak{M}$ be an irredundant PSS model for the language $L = (\mathscr{A}, \Psi)$, and let us make reference to the definitions in Sections 1 and 2.1 and to Definitions 2.4.1 and 2.4.2. Furthermore, let $\mathfrak{M}$ satisfy the SO and SY conditions.

We say that $\mathfrak{M}$ satisfies the condition of *operationality and exactness* (briefly, OE condition) whenever the following statement holds.

OE. Two subsets $\mathscr{E}_O$ and $\mathscr{E}_E$ of $\mathscr{E}$ are defined such that:

(i) For every $\mathbf{E} \in \mathscr{E}_O$, $S_T(\mathbf{E}) \in \mathscr{L}$.

(ii) For every $\mathbf{E} \in \mathscr{E}_E$, $S_T(\mathbf{E}) \in \mathscr{L}$ and $S_F(\mathbf{E}) = S_T^{\perp}(\mathbf{E})$ [hence $S_F(\mathbf{E}) \in \mathscr{L}$].

(iii) $\mathscr{E}_E \subseteq \mathscr{E}_O$.

Whenever $\mathfrak{M}$ satisfies the OE condition, we say that $\mathscr{E}_O$ and $\mathscr{E}_E$ are the sets of symbols of *operational* (or *fuzzy*) and *exact effects*, respectively. In addition, we respectively denote, by abuse of language, by $<$ and $\approx$ the restrictions to $\mathscr{E}_O$, or to $\mathscr{E}_E$, of the binary relations $<$ and $\approx$ defined on $\mathscr{E}$; we also denote by $<$ the order relation canonically induced on $\mathscr{E}_O/\approx$ by the quasi-order $<$ defined on $\mathscr{E}_O$.

By making use of Definition 2.4.3, some further conditions can be required to hold in an irredundant PSS model which formalize properties of operational and exact effects.

Definition 2.4.4. Let $\mathfrak{M}$ be an irredundant PSS model for the language $L = (\mathscr{A}, \Psi)$, and let us make reference to the definitions in Sections 1 and 2.1, and to definitions 2.4.1–2.4.3. Furthermore, let $\mathfrak{M}$ satisfy the SO, SY, and OE conditions.

We say that $\mathfrak{M}$ satisfies the condition of *existence of the complement* (briefly, CE condition) whenever the following statement holds.

CE. For every $\mathbf{E} \in \mathscr{E}_O$, an $\mathbf{E}' \in \mathscr{E}_O$ exists such that, for every $i \in \tilde{I}$, $\rho_i(\mathbf{E}') = D_i \backslash \rho_i(\mathbf{E})$.

We say that $\mathfrak{M}$ satisfies the condition of *exact measurability* (briefly, EM condition) whenever the following statement holds.

EM. for every $\mathbf{E}_1 \in \mathscr{E}_E$, $\mathbf{E}_2 \in \mathscr{E}_O$,

$$\mathbf{E}_1 < \mathbf{E}_2 \quad \text{implies} \quad \text{for every } i \in \tilde{I}, \quad \rho_i(\mathbf{E}_1) \subseteq \rho_i(\mathbf{E}_2)$$

We say that $\mathfrak{M}$ satisfies the condition of *completeness of the exact effects* (briefly, EC condition) whenever the following statement holds.

EC. The restriction S_{TE} of the mapping S_T to $\mathscr{E}_E$ maps surjectively $\mathscr{E}_E$ onto $\mathscr{L}$.

Finally, we say that an irredundant PSS model $\mathfrak{M}$ is a *physical model* whenever it satisfies the SO, SY, OE, CE, EM, and EC conditions.

The following propositions collect some relevant properties of physical models.

Proposition 2.4.1. Let $\mathfrak{M}$ be a physical model for the language $L = (\mathscr{A}, \Psi)$, and let us make reference to the definitions in Sections 1 and 2.1, and to Definitions 2.4.1–2.4.4.

Then, the following statements hold.

(i) For every $\mathbf{E}_1, \mathbf{E}_2 \in \mathscr{E}_E$

$$\mathbf{E}_1 < \mathbf{E}_2 \quad \text{iff} \quad \text{for every } i \in \tilde{I}, \quad \rho_i(\mathbf{E}_1) \subseteq \rho_i(\mathbf{E}_2)$$

(ii) The relation $<$ is a partial order on $\mathscr{E}_E$.

(iii) The restriction S_{TE} of S_T to $\mathscr{E}_E$ is an order isomorphism of $(\mathscr{E}_E, <)$ onto $(\mathscr{L}, \varnothing, \subseteq, {}^\perp)$.

(iv) The poset $(\mathscr{E}_E, <)$ is a complete lattice (with minimal element $\mathbb{0}$ and maximal element $\mathbb{1}$), and the mapping (which we still denote by ${}^\perp$, by abuse of language)

$${}^\perp\!: \quad \mathbf{E} \in \mathscr{E}_E \to \mathbf{E}^\perp = S_{TE}^{-1}(S_{TE}(\mathbf{E})^\perp) \in \mathscr{E}_E$$

is a standard orthocomplementation on $(\mathscr{E}_E, <)$.

(v) The mapping

$${}'\!: \quad \mathbf{E} \in \mathscr{E}_O \to \mathbf{E}' \in \mathscr{E}_O$$

is well defined; moreover, it maps bijectively $\mathscr{E}_O$ onto itself, and its restriction to $\mathscr{E}_E$ (which we still denote by $'$ by abuse of language) coincides with the standard orthocomplementation ${}^\perp$.

Proof. See Garola (1989).

Proposition 2.4.2. Let $\mathfrak{M}$ be a physical model for the language $L = (\mathscr{A}, \Psi)$, and let us make reference to the definitions in Section 1 and 2.1 and to Definitions 2.4.1–2.4.4.

Then, the following statements hold.

(i) For every $\mathbf{E} \in \mathscr{E}_O$, $S_F(\mathbf{E}) \subseteq (S_T(\mathbf{E}))^\perp$ [hence, for every $\mathbf{E}_e \in \mathscr{E}_E$, $\mathbf{E}_e \approx \mathbf{E}$ implies $S_F(\mathbf{E}) \subseteq S_F(\mathbf{E}_e)$].

(ii) For every $\mathbf{E} \in \mathscr{E}_O$, one and only one $\mathbf{E}_e \in \mathscr{E}_E$ exists which is $\approx$-equivalent to $\mathbf{E}$.

(iii) For every $\mathbf{E} \in \mathscr{E}_O$, the mapping

$$\zeta\!: \quad [\mathbf{E}]_\approx \in \mathscr{E}_O/\!\approx \; \to \mathbf{E}_e \in \mathscr{E}_E \cap [\mathbf{E}]_\approx$$

is well defined, and it is an order isomorphism of $(\mathscr{E}_O/\!\approx, <)$ onto $(\mathscr{E}_E, <)$.

(iv) The poset $(\mathscr{E}_O/\!\approx, <)$ is a complete lattice, with minimal element $[\mathbb{0}]_\approx$ and maximal element $[\mathbb{1}]_\approx$, and the mapping (which we still denote by ${}^\perp$, by abuse of language).

$${}^\perp\!: \quad [\mathbf{E}]_\approx \in \mathscr{E}_O/\!\approx \; \to [\mathbf{E}]_\approx^\perp = \zeta^{-1}(\zeta([\mathbf{E}]_\approx)^\perp) \in \mathscr{E}_O/\!\approx$$

is a standard orthocomplementation on $(\mathscr{E}_O/\!\approx, <)$.

(v) Let S_{TO} and S_{TE} be the restrictions of S_T to $\mathscr{E}_O$ and $\mathscr{E}_E$ respectively, and let us put

$$\phi_0: \quad \mathbf{E} \in \mathscr{E}_O \to [\mathbf{E}]_{\approx} \in \mathscr{E}_O/\approx$$

Then, the following diagram (where ζ and S_{TE} are order isomorphisms)

$$\begin{array}{ccc} \mathscr{E}_O & \xrightarrow[S_{TO}]{} & \mathscr{L} \\ \phi_O \downarrow & & \uparrow S_{TE} \\ \mathscr{E}_O/\approx & \xrightarrow[\zeta]{} & \mathscr{E}_E \end{array}$$

is commutative; furthermore, S_{TO} and ϕ_O are surjective and quasi-order preserving.

Proof. See Garola (1989).

2.5. The Intended Interpretation of the Sets $\mathscr{E}_O$ and $\mathscr{E}_E$

The definitions in Section 2.4 restate in our present framework a set of concepts and relations which are standard in some approaches to the foundations of QP (though they also hold in CP). Our present treatment, where the semantic model is explicit and formalized, and every relation between predicates is defined, through the assignment function, by means of relations between sets of physical objects inside laboratories, has the advantage of making the empirical interpretation of the aforesaid concepts and relations more immediate, also providing an intuitive justification for many definitions.

More specifically, let us consider Definition 2.4.1 and let **E** denote an effect which can be actualized by means of a suitable dicotomic apparatus in every laboratory (i.e., which is "operational" in the sense specified in Section 1.3). It follows from the definition of $\mathbf{S}_T$ (via the intended interpretation) that a symbol of state **S** belongs to the certainly yes domain of **E** iff all the physical objects prepared according to the state denoted by **S** give the yes answer in almost every laboratory whenever tested by means of any apparatus which actualizes the effect denoted by **E**. An analogous property characterizes the certainly no domain of **E**, with the no answer in place of the yes answer. Then, the quasi-order relation $<$ and the equivalence relation $\approx$ on $\mathscr{E}$ can be introduced without any explicit reference the extensions of the predicates in $\mathscr{E}$.

Let us come now to condition OE in Definition 2.4.3. The sets $\mathscr{E}_O$ and $\mathscr{E}_E$ are defined by making reference to formal properties of the certainly yes and certainly no domains of their elements, which could be reformulated as metalinguistic schemes of formulas of L having the truth mode of physical laws. We make here a new assumption in our intended interpretation by stating that the symbols of operational effects must be interpreted as nouns of those effects which are "operational" in the sense specified in Section 1.3 (we recall that we do *not* assume in our basic interpretation that *all* effects are operational in this sense). Moreover, every symbol of exact effect is interpreted as a noun of an operational effect that is actualized by an idealized dichotomic device which exactly tests whether the value of a given physical observable lies in a given Borel subset of the real line (briefly, which exactly tests whether a given "physical property" holds[5]).

Bearing in mind this interpretation, the properties in the OE condition can be seen to formalize physical laws that hold both in CP and QP. Furthermore, condition CE in Definition 2.4.4 can immediately be justified; indeed, for every symbol of operational effect $\mathbf{E}$, the symbol of effect $\mathbf{E}'$ can be interpreted as denoting the effect which is actualized in every laboratory by the same apparatus that actualize the effect denoted by $\mathbf{E}$, in each apparatus the yes and no answer being exchanged. For it is apparent that every apparatus of the latter kind will select a set of physical objects in the laboratory $i \in \tilde{I}$ which is the complement of the set of objects selected by any apparatus that actualizes the effect denoted by $\mathbf{E}$.

By making use of the above interpretation, condition EM in Definition 2.4.4 can also be partially justified, since it can be said to express the "fuzziness" of the elements in $\mathscr{E}_O$. Indeed, it implies that for every symbol of effect $\mathbf{E}_2$ which belongs to $\mathscr{E}_O/\mathscr{E}_E$ some physical objects may exist in almost every laboratory which would give the yes answer if tested by an apparatus which actualizes the effect denoted by $\mathbf{E}_2$, while they would give the no answer if tested by an ideal apparatus which actualizes any exact effect, which we denote by $\mathbf{E}_1$, such that $\mathbf{E}_1 < \mathbf{E}_2$ (in particular, such that $\mathbf{E}_1 \approx \mathbf{E}_2$); indeed, this is a reasonable requirement both in CP and in QP.

Finally, the EC condition in Definition 2.4.4, which obviously implies that every $\mathbf{E} \in \mathscr{E}$ such that $S_T(\mathbf{E}) \in \mathscr{L}$ and $S_F(\mathbf{E}) = S_T^{\perp}(\mathbf{E})$ is a symbol of exact effect, can be partially justified by observing that it also implies that, for every $\mathbf{E} \in \mathscr{E}_O/\mathscr{E}_E$, at least one symbol of exact effect $\mathbf{E}_e$ exists which is $\approx$-equivalent to $\mathbf{E}$ and denotes a physical property of which $\mathbf{E}$ denotes a

[5]The minimal effect associated with a state according to the SO condition in Definition 2.1.2 is not necessarily "operational" in the sense specified by Definition 2.4.3; if such a property is required, it must be stated as a further independent condition (see in particular the EO condition in the next subsection); as a matter of fact, this occurs both in CP and in QP. In principle, it is conceivable that the "operativity domain" of a physical theory is defined by means of theoretical objects; this would happen in our framework if the mapping g were such that $g(\mathscr{S}) \not\subseteq \mathscr{E}_O$.

"fuzzy test." Again, this seems a reasonable requirement in most physical theories, where idealized elements are unavoidable (or can be avoided at the expense of understability and manageability of the formal apparatus).

Let us comment now on some results in Propositions 2.4.1 and 2.4.2. It follows from Proposition 2.4.2 that the quasi-order structures on the sets $\mathscr{E}_O$ and $\mathscr{E}_E$, which are defined in Definition 2.4.3, can be interconnected, the connection being summarized by the commutative diagram in statement (v) of Proposition 2.4.2; in the diagram, the posets $(\mathscr{E}_O/\approx, <)$, $(\mathscr{E}_E, <)$, and $(\mathscr{L}, \subseteq)$ are orthocomplemented complete lattices, which are isomorphic. This result has, in our opinion, a great explanatory power, since it allows us to understand the links between known approaches to the foundations of QP [see in particular Mackey (1963), Jauch (1968), Piron (1976), and Ludwig (1983)] and it explains the deep reasons why the same mathematical structure appears in different approaches where different sets of physical concepts are considered primitive. Indeed, it follows from the intended interpretation discussed in Section 1.3 that the set $\mathscr{E}_O$ is interpreted bijectively onto the set of "effects" in Ludwig's sense (but it must be carefully noted that the partial order $\leq$ that can be introduced on $\mathscr{E}_O$ following Ludwig is different from the partial order introduced here; indeed, $\leq$ is stronger than $<$). Furthermore, the set $\mathscr{E}_O/\approx$ can be interpreted bijectively onto the set of "propositions" introduced in the Jauch–Piron approach (in this case, the order $<$ is analogous to the order introduced by Jauch and Piron) and the set $\mathscr{E}_E$ on the set of questions introduced by Mackey (the restrictions to $\mathscr{E}_E$ of $<$ and of Ludwig's order $\leq$ coincide, and both these restrictions coincide with the order that can be directly introduced on $\mathscr{E}_E$ following Mackey; we also notice that the word "question" has a different meaning in Piron's and in Mackey's approaches). Thus, the diagram in statement (v) of Proposition 2.4.2 exhibits the correlations between the fundamental structures of different theoretical approaches [which we have already explored, along with other authors, in the framework of elementary QP; see Garola and Solombrino (1983)]; of course, these are endowed, at this stage, with only those properties which hold both in CP and in QP.

Now let us briefly comment on the epistemological role of the aforesaid structures; indeed, in our opinion, this role has often been misunderstood in the literature. Our above interpretations of $\mathscr{E}_O$ and $\mathscr{E}_E$ may help in making this point clear, as follows.

It is well known that every physical theory determines its own range of "epistemic accessibility"; in our present framework this range is the set of all formulas of L whose truth values can be tested by means of a suitable set of measurements (we reject here the neopositivistic "verification theory of meaning," which leads one to identify meaning and epistemic accessibility). Then, intuitively, an empirical (two-valued) logic is obtained by suitably selecting a subset Ψ_a of formulas in Ψ which are epistemically accessible, by taking these formulas as atomic wffs of a new language L_a, by introducing new connectives in L_a which are defined by (empirical) semantic relations in Ψ_a. A suitable basis for the selection of Ψ_a is usually provided by a set of atomic formulas of L that are themselves epistemically accessible; in particular, by the set $\mathscr{E}_E(x) = \{\mathbf{E}(\mathbf{x}) \in \Psi \mid \mathbf{E} \in \mathscr{E}_E\}$, with $\mathbf{x} \in X$, which can be endowed with the partial order $<$ canonically induced by the order $<$ defined on $\mathscr{E}_E$, so that $(\mathscr{E}_E(\mathbf{x}), <)$ is order isomorphic to $(\mathscr{E}_E, <)$.

The formal languages L_E^x and L_E^S introduced in Section 3, with their induced quasi-order relations $<$, provide concrete examples of the above procedures. In the former case, $\Psi_a = \mathscr{E}_E(\mathbf{x})$; in the latter,

$$\Psi_a = \{\mathbf{A} \in \Psi \mid \mathbf{A} = (\forall \mathbf{x})(\mathbf{S}(\mathbf{x}) \rightarrow \mathbf{E}(\mathbf{x})), \mathbf{x} \in X, \mathbf{S} \in \mathscr{S}, \mathbf{E} \in \mathscr{E}_E\}$$

Let us refer for simplicity to the case $\Psi_a = \mathscr{E}_E(\mathbf{x})$ (the other case can be discussed along similar lines and leads to similar results).

We can consider the Lindenbaum–Tarski algebra $\mathfrak{A}$ of L with the usual "inferential" order $<$; of course, $(\mathfrak{A}, <)$ is a Boolean lattice. Let us select in $\mathfrak{A}$ the subset $\mathfrak{A}_E(\mathbf{x})$ of all the elements which contain a (unique, atomic) formula of $\mathscr{E}_E(\mathbf{x})$.

All the wffs in L that belong to elements of $\mathfrak{A}_E(\mathbf{x})$ are epistemically accessible in the sense specified above (though they do not exhaust the set of all epistemically accessible formulas in L); we say that $\mathfrak{A}_E(\mathbf{x})$ is a set of epistemically accessible propositions of $\mathfrak{A}$. Furthermore, the poset $(\mathfrak{A}_E(\mathbf{x}), <)$ is order isomorphic to $(\mathscr{E}_E(\mathbf{x}), <)$ [the existence of a one-to-one mapping of $\mathfrak{A}_E(\mathbf{x})$ onto $\mathscr{E}_E(\mathbf{x})$ is obvious; we prove in Section 3.1 that the order is preserved by this mapping] and hence, it is an orthocomplemented complete lattice. Yet, its mathematical properties derive from semantic relations between the primitive predicates in $\mathscr{E}_E$ that have the truth mode of physical laws, and not necessarily match the properties of $(\mathfrak{A}, <)$ [in particular, $(\mathfrak{A}_E(\mathbf{x}), <)$ need not be a Boolean lattice], which derive from semantic relations between formulas of L that have the truth mode of logical laws. Thus, we can consider $(\mathscr{E}_E(\mathbf{x}), <)$ [or $(\mathfrak{A}_E(\mathbf{x}), <)$] an "empirical" logic, but it must not be confused with some substructure of the Lindenbaum–Tarski algebra of L, nor can join, meet, and orthocomplementation in it be confused with the operations on $(\mathfrak{A}, <)$ that bear the same name [indeed, the embedding of $(\mathfrak{A}_E(\mathbf{x}), <)$ into $(\mathfrak{A}, <)$ preserves the order; it does not necessarily preserve lattice operations].

This notwithstanding, $(\mathscr{E}_E(x), <)$ [equivalently, $(\mathscr{E}_E, <)$] can be promoted to the role of model for some kind of algebra of propositions of an empirical logic (see Section 3.3); this is usual practice in many approaches to QL, where $(\mathscr{E}_E, <)$ induces an "algebraic" semantics for QL (e.g., Dalla

Chiara, 1977). We must admit our reluctance in considering formal languages with this kind of semantics as full-fledged new logics; our position mainly rests on the above remarks about the truth mode of the semantic relations that support the lattice properties of $(\mathscr{E}_E, <)$.

It is interesting to notice that we will prove in Section 3.4 that our distinction between "empirical" and "logical" structures can be neglected in CP. Intuitively, this occurs because every molecular formula can be associated with an atomic formula in CP which has the same extension in almost every laboratory (consistent with the belief that the truth value of every physical statement can be directly tested by means of a suitable apparatus), so that the (empirical) semantic relations between atomic formulas match the (logical) semantic relations between molecular formulas. The above distinction cannot be ignored in QP, where the "empirical" and the "logical" structures exhibit different mathematical properties.

We note that there are examples, even in QP, of an operation sign which exhibits semantic properties that are similar to those of a suitable logical connective; this has favored confusion between empirical and logical structures (see Garola, 1989).

2.6. Atomicity and Nonpure States

As we have anticipated in the introductory remarks to Section 2, we introduce in the following definition some further conditions on physical models which make the lattice $(\mathscr{E}_E, <)$ atomic and imply that every nonpure state is a mixture in our present framework.

Definition 2.6.1. Let $\mathfrak{M}$ be a physical model for the language $L = (\mathscr{A}, \Psi)$, and let us make reference to the definitions in Sections 1, 2.1, 2.4.

We say that $\mathfrak{M}$ satisfies the condition of *exact observability of states* (briefly, EO condition) whenever the following statement holds.

EO. Let $\mathbf{S} \in \mathscr{S}$; then, $g(\mathbf{S}) \in \mathscr{E}_E$.

We say that $\mathfrak{M}$ satisfies the condition of *invariance of frequency* (briefly, FI condition) iff the following statement holds.

FI. For every $\mathbf{E} \in \mathscr{E}_E$ and $\mathbf{S} \in \mathscr{S}$, a real number $r \in [0, 1]$ exists such that, for every $i \in \tilde{I}$,

$$\frac{\nu_i(\rho_i(\mathbf{E}) \cap \rho_i(\mathbf{S}))}{\nu_i(\rho_i(\mathbf{S}))} = r$$

We say that $\mathfrak{M}$ satisfies the *mixtures condition* (briefly, MS condition) iff the following statement holds.

MS. For every $\mathbf{S} \in \mathscr{S}$, at least one family $(\mathbf{S}_k)_{k \in K}$ (K being a set of indices) of mutually preclusive symbols of pure states exists such that:

(i) For every $i \in \tilde{I}$, $\rho_i(\mathbf{S}) \cap \rho_i(g(\mathbf{S}_k))$ is nonvoid, and

$$\bigcup_{k \in K} (\rho_i(\mathbf{S}) \cap \rho_i(g(\mathbf{S}_k))) = \rho_i(\mathbf{S})$$

(ii) For every $i \in \tilde{I}$ and $\mathbf{E} \in \mathscr{E}_E$,

$$\frac{\nu_i(\rho_i(\mathbf{S}) \cap \rho_i(g(\mathbf{S}_k)) \cap \rho_i(\mathbf{E}))}{\nu_i(\rho_i(\mathbf{S}) \cap \rho_i(g(\mathbf{S}_k)))} = \frac{\nu_i(\rho_i(\mathbf{S}_k) \cap \rho_i(\mathbf{E}))}{\nu_i(\rho_i(\mathbf{S}_k))}$$

Finally, let $\mathfrak{M}$ satisfy the EO, FI, and MS conditions; then, for every $\mathbf{S} \in \mathscr{S}$ we say that $\mathbf{S}$ denotes a mixture whenever $\mathbf{S} \in \mathscr{S} \backslash \mathscr{S}_P$.

The following propositions collect some properties of physical models that satisfy the EO, FI, and MS conditions.

Proposition 2.6.1. Let $\mathfrak{M}$ be a physical model for the language $L = (\mathscr{A}, \Psi)$, and let us make reference to the definitions in Sections 1, 2.1, and 2.4 and to Definition 2.6.1.

Let $\mathfrak{M}$ satisfy the EO condition. Then the lattice $(\mathscr{L}, \varnothing, \subseteq, ^\perp)$ is atomic, and the set of its atoms is $\{\{\mathbf{S}\} \mid \mathbf{S} \in \mathscr{S}_P\}$ (hence, also the lattices $(\mathscr{E}_O/\approx, [\mathbb{O}]_\approx, <, ^\perp)$ and $(\mathscr{E}_E, \mathbb{O}, <, ')$ are atomic, and the sets of their atoms are $\{[g(\mathbf{S})]_\approx \mid \mathbf{S} \in \mathscr{S}_P\}$ and $\{g(\mathbf{S}) \mid S \in \mathscr{S}_P\}$, respectively).

Proof. See Garola (1989).

Proposition 2.6.2. Let $\mathfrak{M}$ be a physical model for the language $L = (\mathscr{A}, \Psi)$, and let us make reference to the definitions in Sections 1, 2.1, and 2.4 and to Definition 2.6.1.

Let $\mathfrak{M}$ satisfy the EO, FI, and MS conditions, and for every $\mathbf{S} \in \mathscr{S}$, $k \in K$, $\mathbf{E} \in \mathscr{E}_E$ let us put

$$\lambda_k = \frac{\nu_i(\rho_i(\mathbf{S}) \cap \rho_i(g(\mathbf{S}_k)))}{\nu_i(\rho_i(\mathbf{S}))}$$

$$r = \frac{\nu_i(\rho_i(\mathbf{S}) \cap \rho_i(\mathbf{E}))}{\nu_i(\rho_i(\mathbf{S}))}$$

$$r_k = \frac{\nu_i(\rho_i(\mathbf{S}_k) \cap \rho_i(\mathbf{E}))}{\nu_i(\rho_i(\mathbf{S}_k))}$$

Then, whenever the set K of indices in the MS condition is denumerable, the following statements hold.

(i) $r = \sum_{k \in K} \lambda_k r_k$.

(ii) $\sum_{k \in K} \lambda_k = 1$.

(iii) For every $k \in K$, $\mathbf{S}_k \in S_T(g(\mathbf{S}))$

Proof. See Garola (1989).

2.7. The Intended Interpretation of the EO, FI, and MS Conditions

The conditions introduced in Definition 2.6.1 can be interpreted and physically justified by making use of the intended physical interpretation treated in Sections 1.3 and 2.5.

Let us begin with the EO condition. Bearing in mind our interpretation of symbols of exact effects in Section 2.5, this condition formalizes the requirement that the minimal effect associated with every state by the SO condition in Definition 2.1.1 be an exact effect. This requirement sounds physically reasonable, since this minimal effect can be interpreted, loosely speaking, as the better possible test of whether a physical object is in the given state. In addition, condition EO is strongly supported by its consequences, since it implies that in every physical model $\mathfrak{M}$ the lattice $(\mathscr{E}_E, <)$ is atomic (see Proposition 1.7.1); indeed, atomicity is a fundamental property of the basic lattices introduced in many approaches to the foundations of physics (see in particular the approaches quoted in Section 2.5), which are isomorphic to $(\mathscr{E}_E, \mathbb{O}, <, ')$.

Let us come now to the FI condition in Definition 2.6.1. According to our intended interpretation, this condition formalizes the requirement that an exact effect selects physical objects prepared according to a given state with the same frequency in (almost) every laboratory. Now, according to classical and quantum physics, a real number $r \in [0, 1]$ exists for every state and every exact effect which expresses the probability that a physical object prepared according to the given state be selected by the given exact effect; therefore, the FI condition formalizes this probabilistic physical law in statistical approximation (in the sense specified in Section 1.8).

Finally, let us consider the MS condition. Here, statements (i) and (ii) formalize, via our intended interpretation, two assumptions about nonpure states that are usual both in CP and in QP. To be precise, statement (i) says that for every state denoted by $\mathbf{S}$, some family of mutually preclusive pure states exists such that, in (almost) every laboratory i, every physical object in $\rho_i(\mathbf{S})$ would be selected by the minimal exact effect (more explicitly, by some apparatus which actualizes in i the minimal exact effect) associated with a suitable pure state in the family. Whenever the family is unique, as in CP (see Section 3.4), this statement translates in our present framework the familiar assumption that every physical object prepared according to a nonpure state actually "is in some pure state" [since $\rho_i(\mathscr{S})$ is a partition of D_i in our present approach, these words cannot be endowed here with their standard set-theoretic meaning]. Whenever the family is not necessarily unique, as in QP, then for every mixture one privileged family can be chosen that admits the same interpretation above [yet it is well known that the existence of many equivalent mathematical representations of the same mixture in QP is a source of difficulties in the interpretation of the theory (e.g., Beltrametti and Cassinelli, 1981)]. Then, statement (ii) says that, if one takes into account those physical objects in $\rho_i(\mathbf{S})$ that "are in the pure state denoted by $\mathbf{S}_k$" (in the sense specified above) and evaluates the percentage of these that would be selected by a given exact effect, one finds the same value obtained whenever the percentage is evaluated of individual systems in $\rho_i(\mathbf{S}_k)$ that would be selected by the same effect. This is an intuitive and known physical law if statistical percentages are substituted with selection probabilities; therefore, our statement (ii) formalizes a probabilistic physical law in the statistical approximation.

2.8. Some Remarks on Entities

We intend to discuss briefly in the present section the concept of "physical system" because of its central role in CP and in QP; however, we do not intend to explore it in detail, since it has no relevance in the sequel.

Following other authors (e.g., Aerts, 1982; Foulis *et al.*, 1983), we often use the word "entity" in place of "physical system" here; this avoids in particular any confusion with the locution "physical object" that has been used throughout our paper with a different meaning.

Now, let us observe that in common physical language every entity can be identified with the set of all properties that characterize it; thus, by making use of our intended interpretation (see in particular Section 2.5), an entity can be defined by means of a suitable family of exact effects. Yet not every family of this kind defines an entity; indeed, some conditions must be imposed which derive (via the intended interpretation) from the standard physical concept of entity.

In order to state these conditions, we need some further definitions. Therefore, let $\mathfrak{M}$ be a physical model for the language $L=(\mathscr{A},\Psi)$. Then, for every family $\mathbf{R}=(\mathbf{E}_k)_{k\in K}$ (K being a set of indices) of symbols of exact effects, and for every laboratory $i\in I$, we call the *extension* of $\mathbf{R}$ in i the set $\rho_i(\mathbf{R})$ of physical objects defined as follows:

$$\rho_i(\mathbf{R})=\bigcap_{k\in K}\rho_i(\mathbf{E}_k)$$

In addition, for every predicate $\mathbf{P}\in\mathscr{P}$ we say that $\mathbf{P}$ *belongs to* $\mathbf{R}$, and write $\mathbf{P}\subset\mathbf{R}$, whenever the following relation holds

$$\text{for every } i\in\tilde{I},\quad \rho_i(\mathbf{P})\subseteq\rho_i(\mathbf{R})$$

and denote by $\mathscr{S}_R$ the set of all symbols of state that belong to $\mathbf{R}$.

Now, we say that the family $\mathbf{R} = (\mathbf{E}_k)_{k \in K}$ defines an entity, which we still denote by $\mathbf{R}$, whenever the poset $(g(\mathcal{S}_R), <)$ is a complete sublattice of $(\mathscr{E}_E, <)$, orthocomplemented by the restriction of $^\perp$ to $g(\mathcal{S}_R)$, the maximal element $\mathbb{1}_R$ of it being such that, for every $i \in \tilde{I}$, $\rho_i(\mathbb{1}_R) = \rho_i(\mathbf{R})$.

Let us consider now some consequences of the above definition.

First, we note explicitly that at least one entity is defined for every physical model $\mathfrak{M}$ if we assume that for every $\mathbf{E} \in \mathscr{E}_E$ some $\mathbf{S} \in \mathcal{S}$ exists such that $\mathbf{E} = g(\mathbf{S})$—more precisely, the entity defined by the family $\mathbf{R} = \{\mathbb{1}\}$.

Second, observe that, for every entity denoted by $\mathbf{R}$ and for every $\mathbf{S} \in \mathcal{S}_R$, we get $g(\mathbf{S}) \subset \mathbf{R}$; indeed, $g(\mathbf{S}) < \mathbb{1}_R$, and hence, for every $i \in \tilde{I}$, $\rho_i(g(\mathbf{S})) \subseteq \rho_i(\mathbb{1}_R) = \rho_i(\mathbf{R})$ because of the EM condition in Definition 2.4.4. This result can be interpreted, via the intended interpretation, as saying that every minimal effect which tests a state of some given entity selects only samples of this entity.

Third, for every entity denoted by $\mathbf{R}$, let us consider the maximal effect denoted by $\mathbb{1}_R$. Since $\rho_i(\mathbb{1}_R) = \rho_i(\mathbf{R})$, it follows that samples of this entity can be recognized with certainty, because the exact effect denoted by $\mathbb{1}_R$ selects in every laboratory those, and only those, individual systems which are samples of it.

Fourth, let us assume that the EO, FI, and MS conditions in Definition 2.6.1 hold, and let us introduce the further assumption that the set $\mathcal{S}$ can be partitioned into subsets of symbols of states that belong to different entities; let us denote by $\mathbf{R}$ one of these entities, and let $\mathbf{S}$ denote a nonpure state and be such that $\mathbf{S} \subset \mathbf{R}$. Then, it is immediate to prove that $\mathbf{S}$ can be decomposed (in the sense specified by the MS condition) by means of symbols or pure states that belong to $\mathbf{R}$.

3. OPERATIONAL LOGICS

We introduce in the present section (Sections 3.1–3.3) two new elementary formal languages L_E^x and L_E^S, which are constructed for making statements about a physical object $\mathbf{x}$ and about the set of physical objects prepared according to the state denoted by $\mathbf{S}$, respectively. We show that the sets Ψ_E^x and Ψ_E^S of all formulas of L_E^x and L_E^S, respectively, can be mapped into the set Ψ of wffs of L in such a way that the semantic properties induced in both languages by the properties that hold in L coincide with those which hold in known two-valued quantum logical structures, while the induced interpretations coincide with different admissible interpretations in QL (the syntactic differences between L_E^x and L_E^S being the counterpart of the differences in the interpretations). Thus, we explain the (logical) genesis of quantum logical structures in classical terms. In addition, we show that also "fuzzy logics" can be derived and interpreted by means of similar techniques in our classical framework.

In Section 3.4 we compare CP and QP from our present viewpoint, and briefly discuss some epistemologically relevant features of our approach.

3.1. The Individual Language L_E^x

With the following definitions, we introduce, for every $\mathbf{x} \in X$, a formal language L_E^x, the statements of which concern a given physical object $\rho_i^\sigma(\mathbf{x})$ whenever an interpretation of the variables σ and a laboratory i are specified.

Definition 3.1.1. Let $\mathfrak{M}$ be a physical model for the language $L = (\mathscr{A}, \Psi)$, and let us make reference to the definitions in Sections 1 and 2.

Let $\mathbf{x}$ be interpreted on an individual variable in X. We denote by $\mathscr{A}_E^x$ the set of descriptive, logical, and auxiliary signs defined as follows.

Descriptive signs

D1. Individual signs: the variable in X denoted by $\mathbf{x}$.

D2. Predicative signs: all the monadic predicates in $\mathscr{E}_E$.

Logical signs

L1. Signs of connectives: $^\perp$, $\sqcap$, $\sqcup$.

Auxiliary signs

A1. Round parentheses $(\cdot)$.

Furthermore, we denote by Ψ_E^x the set of all well-formed formulas constructed by means of the signs in $\mathscr{A}_E^x$ and of the following rules:

W1. For every $\mathbf{E} \in \mathscr{E}_E$, $\mathbf{E}(\mathbf{x}) \in \Psi_E^x$.

W2. For every $\mathbf{A}(\mathbf{x}) \in \Psi_E^x$, $\mathbf{A}^\perp(\mathbf{x}) \in \Psi_E^x$.

W3. for every $\mathbf{A}(\mathbf{x})$, $\mathbf{B}(\mathbf{x}) \in \Psi_E^x$,

$$\mathbf{A}(\mathbf{x}) \sqcap \mathbf{B}(\mathbf{x}) \in \Psi_E^x \quad \text{and} \quad \mathbf{A}(\mathbf{x}) \sqcup \mathbf{B}(\mathbf{x}) \in \Psi_E^x$$

Finally, we call a *formal language* L_E^x the pair $L_E^x = (\mathscr{A}_E^x, \Psi_E^x)$ and denote by $\mathscr{E}_E(\mathbf{x})$ the set of all atomic wffs in Ψ_E^x (which trivially coincides with $\Psi_E^x \cap \Psi$).

We introduce canonical mappings of L_E^x onto $\mathscr{E}$ and into L and define a truth function on Ψ_E^x by means of the truth function defined on L, as follows.

Definition 3.1.2. Let $\mathfrak{M}$ be a physical model for the language $L = (\mathscr{A}, \Psi)$, and let us make reference to the definitions in Sections 1 and 2 and to Definition 3.1.1.

Bearing in mind Proposition 2.4.1 (iv), we denote (by abuse of language) by $\sqcap$ and $\sqcup$ the join and meet, respectively, in the lattice $(\mathscr{E}_E, \mathbb{0}, <, ^\perp)$. Furthermore, for every $\mathbf{x} \in X$, we denote by ω^x the mapping

$$\omega^x: \quad \mathbf{A}(\mathbf{x}) \in \Psi_E^x \to \mathbf{E}_A \in \mathscr{E}_E$$

recursively defined as follows:

$$\text{for every } \mathbf{E} \in \mathscr{E}_E, \qquad \omega^x(\mathbf{E}(\mathbf{x})) = \mathbf{E}$$
$$\text{for every } \mathbf{A}(\mathbf{x}) \in \Psi_E^x, \qquad \omega^x(\mathbf{A}^\perp(\mathbf{x})) = (\mathbf{E}_A)^\perp$$
$$\text{for every } \mathbf{A}(\mathbf{x}), \mathbf{B}(\mathbf{x}) \in \Psi_E^x, \qquad \omega^x(\mathbf{A}(\mathbf{x}) \cap \mathbf{B}(\mathbf{x})) = \mathbf{E}_A \cap \mathbf{E}_B$$
$$\omega^x(\mathbf{A}(\mathbf{x}) \cup \mathbf{B}(\mathbf{x})) = \mathbf{E}_A \cup \mathbf{E}_B$$

and denote (by abuse of language) by $<$ the quasi-order on Ψ_E^x defined as follows:

$$\text{for every } \mathbf{A}(\mathbf{x}), \mathbf{B}(\mathbf{x}) \in \Psi_E^x, \quad \mathbf{A}(\mathbf{x}) < \mathbf{B}(\mathbf{x}) \quad \text{iff} \quad \mathbf{E}_A < \mathbf{E}_B$$

Consequently, we denote (by abuse of language) by $\approx$ and $<$, respectively, the equivalence on Ψ_E^x and the order on $\Psi_E^x/\approx$ canonically induced by the quasi-order $<$ defined on Ψ_E^x.

Furthermore, we call the *canonical translation mapping from L_E^x into L* the mapping (which is identical on the set of the atomic wffs of L_E^x)

$$\tau^x: \quad \mathbf{A}(\mathbf{x}) \in \Psi_E^x \to \mathbf{E}_A(\mathbf{x}) \in \mathscr{E}_E(\mathbf{x}) = \Psi_E^x \cap \Psi$$

Finally, for every $i \in I$, $\sigma \in \Sigma$, we call the *truth function on* Ψ_E^x the mapping

$$f_{i\sigma}^x: \quad \mathbf{A}(\mathbf{x}) \in \Psi_E^x \to f_{i\sigma}(\mathbf{E}_A(\mathbf{x})) \in \{0, 1\}$$

The following proposition collects some fundamental properties of the structure $(\Psi_E^x/\approx, <)$ and of the truth function $f_{i\sigma}^x$ introduced in Definition 3.1.2.

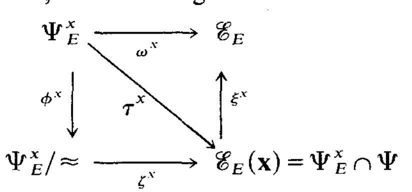
Proposition 3.1.1. Let $\mathfrak{M}$ be a physical model for the language $L = (\mathscr{A}, \Psi)$, and let us make reference to the definitions in Sections 1 and 2 and to Definitions 3.1.1 and 3.1.2.

Let $\mathbf{x} \in X$. Then, the following statements hold.

(i) Let us introduce the mappings

$$\phi^x: \quad \mathbf{A}(\mathbf{x}) \in \Psi_E^x \to [\mathbf{A}(\mathbf{x})]_\approx \in \Psi_E^x/\approx$$
$$\zeta^x: \quad [\mathbf{A}(\mathbf{x})]_\approx \in \Psi_E^x/\approx \to \mathbf{E}_A(\mathbf{x}) \in \mathscr{E}_E(\mathbf{x})$$
$$\xi^x: \quad \mathbf{E}(\mathbf{x}) \in \mathscr{E}_E(\mathbf{x}) \to \mathbf{E} \in \mathscr{E}_E$$

Then, ζ^x is well defined, and the diagram



is commutative. Furthermore, ϕ^x, τ^x, and ω^x are surjective and preserve the quasi-order, while ζ^x and ξ^x are order isomorphisms; hence, $(\mathscr{E}_E(\mathbf{x}), <)$

and $(\Psi_E^x/\approx, <)$ are complete lattices, with minimal elements $\mathbb{O}(\mathbf{x})$ and $[\mathbb{O}(\mathbf{x})]_\approx$, respectively, orthocomplemented by the well-defined mappings, both denoted by $^\perp$, by abuse of language,

$$^\perp\!: \quad \mathbf{E}(\mathbf{x}) \in \mathscr{E}_E(\mathbf{x}) \to \mathbf{E}^\perp(\mathbf{x}) \in \mathscr{E}_E(\mathbf{x})$$

$$^\perp\!: \quad [\mathbf{A}(\mathbf{x})]_\approx \in \Psi_E^x/\approx \,\to [\mathbf{A}^\perp(\mathbf{x})]_\approx \in \Psi_E^x/\approx$$

(ii) Let $\mathbf{A}(\mathbf{x})$, $\mathbf{B}(\mathbf{x}) \in \Psi_E^x$; then,

$$\mathbf{A}(\mathbf{x}) < \mathbf{B}(\mathbf{x}) \quad \text{iff} \quad \text{for every } i \in \tilde{I} \text{ and } \sigma \in \Sigma, \quad f_{i\sigma}^x(\mathbf{A}(\mathbf{x})) \leq f_{i\sigma}^x(\mathbf{B}(\mathbf{x}))$$

[hence, $\mathbf{A}(\mathbf{x}) \approx \mathbf{B}(\mathbf{x})$ iff for every $i \in \tilde{I}$ and $\sigma \in \Sigma$, $f_{i\sigma}^x(\mathbf{A}(\mathbf{x})) = f_{i\sigma}^x(\mathbf{B}(\mathbf{x}))$].

Proof. See Garola (1989).

3.2. The State Language L_E^S

With the following definitions we introduce, for every $\mathbf{S} \in \mathscr{S}$, a formal language L_E^S, the statements of which regard a set of physical objects $\rho_i(\mathbf{S})$ whenever a laboratory i is specified.

Definition 3.2.1. Let $\mathfrak{M}$ be a physical model for the language $L = (\mathscr{A}, \Psi)$, and let us make reference to the definitions in Sections 1 and 2.

Let $\mathbf{S}$ be interpreted on a state symbol in $\mathscr{S}$. We denote by $\mathscr{A}_E^S$ the set of descriptive, specific, logical, and auxiliary signs defined as follows.

Descriptive signs

D1. Predicative signs of kind $\mathscr{E}$, type 1: all the monadic predicates in $\mathscr{E}_E$.

D2. Predicative signs of kind $\mathscr{S}$, type 1: the symbol of state in $\mathscr{S}$ denoted by $\mathbf{S}$.

Specific signs

S1. Diadic predicative constant of type 2: $\subset$.

Logical signs

L1. Signs of connectives $^\perp$, $\cap$, $\cup$.

Auxiliary signs

A1. Comma ,; round parentheses $(\cdot)$.

Furthermore, we denote by Ψ_E^S the set of all well-formed formulas constructed by means of the signs in $\mathscr{A}_E^S$ and of the following rules.

W1. For every $\mathbf{E} \in \mathscr{E}_E$, $\subset(\mathbf{S}, \mathbf{E}) \in \Psi_E^S$.

W2. For every $\mathbf{A}(\mathbf{S}) \in \Psi_E^S$, $\mathbf{A}^\perp(\mathbf{S}) \in \Psi_E^S$.

W3. For every $\mathbf{A}(\mathbf{S})$, $\mathbf{B}(\mathbf{S}) \in \Psi_E^S$,

$$\mathbf{A}(\mathbf{S}) \cap \mathbf{B}(\mathbf{S}) \in \Psi_E^S \quad \text{and} \quad \mathbf{A}(\mathbf{S}) \cup \mathbf{B}(\mathbf{S}) \in \Psi_E^S$$

Finally, we call the *formal language* L_E^S the pair $L_E^S = (\mathscr{A}_E^S, \Psi_E^S)$.

We introduce canonical mappings of L_E^S onto $\mathscr{E}_E$ and into L and define a truth function and a fuzzy-truth function on Ψ_E^S by means of the two-valued truth function defined on L, as follows.

Definition 3.2.2. Let $\mathfrak{M}$ be a physical model for the language $L = (\mathscr{A}, \Psi)$, and let us make reference to the definitions in Sections 1 and 2 and to Definition 3.2.1.

Let $\mathbf{S} \in \mathscr{S}$; we denote by ω^S the mapping

$$\omega^S: \quad \mathbf{A}(\mathbf{S}) \in \Psi_E^S \to \mathbf{E}^A \in \mathscr{E}_E$$

recursively defined as follows:

$$\begin{array}{ll} \text{for every } \mathbf{E} \in \mathscr{E}_E, & \omega^S(\subset(\mathbf{S}, \mathbf{E})) = \mathbf{E} \\ \text{for every } \mathbf{A}(\mathbf{S}) \in \Psi_E^S, & \omega^S(\mathbf{A}^\perp(\mathbf{S})) = (\mathbf{E}^A)^\perp \\ \text{for every } \mathbf{A}(\mathbf{S}), \mathbf{B}(\mathbf{S}) \in \Psi_E^S, & \omega^S(\mathbf{A}(\mathbf{S}) \cap \mathbf{B}(\mathbf{S})) = \mathbf{E}^A \cap \mathbf{E}^B \\ & \omega^S(\mathbf{A}(\mathbf{S}) \cup \mathbf{B}(\mathbf{S})) = \mathbf{E}^A \cup \mathbf{E}^B \end{array}$$

and denote (by abuse of language) by $<$ the quasi-order on Ψ_E^S defined as follows:

$$\text{for every } \mathbf{A}(\mathbf{S}), \mathbf{B}(\mathbf{S}) \in \Psi_E^S, \quad \mathbf{A}(\mathbf{S}) < \mathbf{B}(\mathbf{S}) \quad \text{iff} \quad \mathbf{E}^A < \mathbf{E}^B$$

Consequently, we denote (by abuse of language) by $\approx$ and $<$, respectively, the equivalence on Ψ_E^S and the order on $\Psi_E^S/\approx$ canonically induced by the quasi-order $<$ defined on Ψ_E^S.

We call the *canonical translation mapping* from L_E^S into L the mapping

$$\tau^S: \quad \mathbf{A}(\mathbf{S}) \in \Psi_E^S \to (\forall \mathbf{x})(\mathbf{S}(\mathbf{x}) \to \mathbf{E}^A(\mathbf{x})) \in \Psi$$

with $\mathbf{x}$ any variable in X.

Bearing in mind Proposition 1.6.2(v), for every $i \in I$ we call the *truth function on* Ψ_E^S the mapping

$$f_i^S: \quad \mathbf{A}(\mathbf{S}) \in \Psi_E^S \to f_{i\sigma}((\forall \mathbf{x})(\mathbf{S}(\mathbf{x}) \to \mathbf{E}^A(\mathbf{x}))) \in \{0, 1\}$$

with σ any interpretation of the variables in Σ.

We call the *family of fuzzy translation mappings from* L_E^S *into* L the family $(\vartheta_r^S)_{r \in [0,1]}$, where

$$\vartheta_r^S: \quad \mathbf{A}(\mathbf{S}) \in \Psi_E^S \to (\pi_r \mathbf{x})\mathbf{E}^A(\mathbf{x})/\mathbf{S}(\mathbf{x}) \in \Psi$$

with $\mathbf{x}$ any variable in X.

Bearing in mind Proposition 1.6.2(i), for every $i \in I$ we call the *fuzzy-truth function on* Ψ_E^S the mapping

$$\varphi_i^S: \quad \mathbf{A}(\mathbf{S}) \in \Psi_E^S \to r_i \in [0, 1]$$

where $\varphi_i^S(\mathbf{A}(\mathbf{S})) = r_i$ is the unique real number in $[0, 1]$ such that, for any $\sigma \in \Sigma$, $f_{i\sigma}((\pi_{r_i}\mathbf{x})\mathbf{E}^A(\mathbf{x})/\mathbf{S}(\mathbf{x})) = 1$.

The following proposition collects some fundamental properties of the structure $(\Psi_E^S/\approx, <)$, of the truth function f_i^S, and of the fuzzy-truth function φ_i^S introduced in Definition 3.2.2.

Proposition 3.2.1. Let $\mathfrak{M}$ be a physical model for the language $L = (\mathscr{A}, \Psi)$, and let us make reference to the definitions in Sections 1 and 2 and to Definitions 3.2.1 and 3.2.2.

Let $\mathbf{S} \in \mathscr{S}$. Then, the following statements hold.

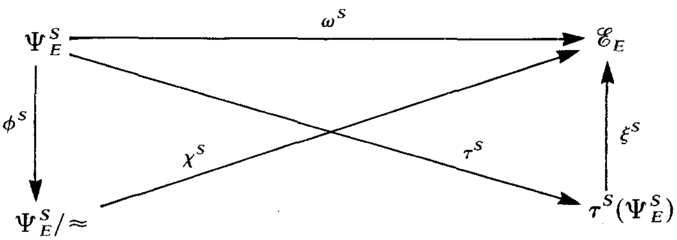
(i) Let us introduce the mappings

$$\phi^S: \quad \mathbf{A}(\mathbf{S}) \in \Psi_E^S \to [\mathbf{A}(\mathbf{S})]_\approx \in \Psi_E^S/\approx$$

$$\chi^S: \quad [\mathbf{A}(\mathbf{S})]_\approx \in \Psi_E^S/\approx \to \mathbf{E}^A \in \mathscr{E}_E$$

$$\xi^S: \quad (\forall \mathbf{x})(\mathbf{S}(\mathbf{x}) \to \mathbf{E}(\mathbf{x})) \in \tau^S(\Psi_E^S) \to \mathbf{E} \in \mathscr{E}_E$$

Then χ^S is well defined and the diagram



is commutative. Furthermore, ω^S and ϕ^S are surjective and preserve the quasi-order, while χ^S is an order isomorphism; hence, the partially ordered set $(\Psi_E^S/\approx, <)$ is a complete lattice, with minimal element $[\subset(\mathbf{S}, \mathbb{0})]_\approx$, orthocomplemented by the well-defined mapping, still denoted by $^\perp$, by abuse of language,

$$^\perp: \quad [\mathbf{A}(\mathbf{S})]_\approx \in \Psi_E^S/\approx \to [\mathbf{A}^\perp(\mathbf{S})]_\approx \in \Psi_E^S/\approx$$

(ii) Let $\mathbf{A}(\mathbf{S}), \mathbf{B}(\mathbf{S}) \in \Psi_E^S$; then,

$\mathbf{A}(\mathbf{S}) < \mathbf{B}(\mathbf{S})$ iff

for every $i \in \tilde{I}$ and $\mathbf{S}^* \in \mathscr{S}$, $f_i^{S^*}(\mathbf{A}(\mathbf{S})) \le f_i^{S^*}(\mathbf{B}(\mathbf{S}^*))$ iff

for every $i \in \tilde{I}$ and $\mathbf{S}^* \in \mathscr{S}$, $\varphi_i^{S^*}(\mathbf{A}(\mathbf{S}^*)) \le \varphi_i^{S^*}(\mathbf{B}(\mathbf{S}^*))$

[Hence,

$\mathbf{A}(\mathbf{S}) \approx \mathbf{B}(\mathbf{S})$ iff

for every $i \in \tilde{I}$ and $\mathbf{S}^* \in \mathscr{S}$, $f_i^{S^*}(\mathbf{A}(\mathbf{S}^*)) = f_i^{S^*}(\mathbf{B}(\mathbf{S}^*))$ iff

for every $i \in \tilde{I}$ and $\mathbf{S}^* \in \mathscr{S}$, $\varphi_i^{S^*}(\mathbf{A}(\mathbf{S}^*)) = \varphi_i^{S^*}(\mathbf{B}(\mathbf{S}^*))$]

(iii) Whenever $\mathfrak{M}$ satisfies the FI condition, the mappings f_i^S and φ_i^S are independent of *i*.

Proof. See Garola (1989).

3.3. The Derived Intended Interpretation of L_E^x and L_E^S

We comment in the present section on the languages L_E^x and L_E^S in the framework of the intended physical interpretation introduced in Sections 1.3, 1.5, and 2.5.

Let us begin with the interpretation of L_E^x. Let $\mathbf{A}(\mathbf{x}) \in \Psi_E^x$. Then, the mapping ω^x in Definition 3.2.1 associates with $\mathbf{A}(\mathbf{x})$ a symbol of exact effect $\mathbf{E}_A \in \mathscr{E}_E$, and the canonical translation mapping τ^x maps $\mathbf{A}(\mathbf{x})$ on the atomic wff $\mathrm{E}_A(\mathbf{x})$; the latter belongs to $\Psi_E^x \cap \Psi$, so that it can be interpreted, according to the rules in Section 1.5, enriched with the interpretation of $\mathscr{E}_E$ discussed in Section 2.5, as follows:

"The physical object denoted by **x** has the physical property denoted by $\mathbf{E}_A$."

This interpretation will be adopted even for the wff $\mathbf{A}(\mathbf{x}) \in \Psi_E^x$.

Let us come now to the quasi-order structure $(\Psi_E^x, <)$. As observed at the end of Section 2.5, the formal language L_E^x, with the induced relation $<$ [hence, more specifically, $(\Psi_E^x, <)$], provides an instance of a general procedure which can be used in order to construct "empirical logics;" indeed, it is obtained by selecting the subset $\mathscr{E}_E(\mathbf{x})$ of epistemically accessible formulas of L and by introducing new symbols of operation in this set whose semantic properties follow from semantic relations between wffs in $\mathscr{E}_E(\mathbf{x})$. Thus, $(\Psi_E^x, <)$ can be considered from two different viewpoints, the first one arising from the definition of $<$ and from statement (i) in Proposition 3.1.1, and the second one arising from statement (ii) in Proposition 3.1.1. According to the former, the lattice structure of $(\Psi_E^x/\approx, <)$, which is order-isomorphic to $(\mathscr{E}_E, <)$, turns out to be a consequence of formal conditions on the model $\mathfrak{M}$ which express semantic relations between primitive predicates and have the truth mode of physical laws, so that it simply reflects the empirical structure of the set of physical properties. According to the latter, the quasi-order relation $<$ on Ψ_E^x can be identified with the quasi-order induced on Ψ_E^x by the family of truth functions $\{f_{i\sigma}^x\}_{i \in I, \sigma \in \Sigma}$; in this sense, $(\Psi_E^x, <)$ can be considered a "logical" structure, and $(\Psi_E^x/\approx, <)$ its algebra of propositions [since every $f_{i\sigma}^x$ is completely determined by the restriction of $f_{i\sigma}$ to the subset $\mathscr{E}_E(\mathbf{x})$ of atomic wffs of

L, or *contingent* part of $f_{i\sigma}$, which is not concerned with the logical apparatus of L, $(\Psi_E^x, <)$ maintains an empirical character even if this perspective is adopted]. We refer the reader to Section 2.5 for further comments on this metatheoretical attitude.

No matter what viewpoint is embraced, the structure $(\psi_E^x/\approx, <)$, which occurs both in CP and in QP, exhibits the basic semantic properties usually assumed when an algebraic semantics for QL is assigned (it is a complete orthocomplemented, possibly atomic, lattice); because of the aforesaid isomorphism with $(\mathscr{E}_E, <)$, this structure can be endowed with further mathematical properties by means of further physical assumptions on $(\mathscr{E}_E, <)$ (in particular, it is distributive in CP, and weakly modular and satisfying the covering law in QP). In addition, the above interpretation of Ψ_E^x is a possible interpretation in QL. Thus, we can affirm that we have derived a quantum logical structure in our extended classical framework on the basis of extralogical (more precisely, physical) assumptions; of course, such a derivation suggests that considering $(\Psi_E^x, <)$ as a "logic" in the full sense is philosophically questionable.

In any case, it is important to note that the connectives $^\perp$, $\sqcap$, and $\sqcup$ in the alphabet of L_E^x cannot be *a priori* identified with the connectives $\neg$, $\wedge$, and $\vee$, respectively, in the alphabet of L (though the identification can be made in CP). Rather, their interpretation must be deduced from the above interpretation of the wffs in Ψ_E^x [for instance, for every $\mathbf{E}_1, \mathbf{E}_2 \in \mathscr{E}_E$, $\mathbf{E}_1(\mathbf{x}) \sqcup \mathbf{E}_2(\mathbf{x})$ states that the physical object denoted by $\mathbf{x}$ has the property denoted by $\mathbf{E}_1 \sqcup \mathbf{E}_2$]. As a matter of fact, the identification between $^\perp$ and $\neg$ turns out to be physically possible in this case (Garola, 1989).

Finally, we stress that any truth function $f_{i\sigma}^x$ is two-valued, as we expect if it is interpreted as attributing a truth value to statements about physical objects.

Let us discuss now the interpretations of L_E^S. We have a unique mapping ω^S in Definition 3.2.2 that associates with every $\mathbf{A}(\mathbf{S}) \in \Psi_E^S$ a symbol of exact effect $\mathbf{E}^A \in \mathscr{E}_E$; yet, we define two different kinds of translations of L_E^S into L, since the canonical translation mapping τ^S and the family of fuzzy translation mappings $(\vartheta_r^S)_{r \in [0,1]}$ are introduced. Therefore, different interpretations of L_E^S can be obtained in the two cases, and we deal with them separately.

We begin with the canonical translation mapping. According to τ^S, any wff $\mathbf{A}(\mathbf{S})$ that belongs to Ψ_E^S is mapped on the wff $(\forall \mathbf{x})(\mathbf{S}(\mathbf{x}) \rightarrow \mathbf{E}^A(\mathbf{x}))$ that belongs to Ψ and can be interpreted, according to the rules in Sections 1.5 and 2.5, as follows:

"All physical objects prepared according to the state denoted by $\mathbf{S}$ have the physical property denoted by $\mathbf{E}^A$."

This interpretation will be adopted even for the wff $\mathbf{A}(\mathbf{S}) \in \Psi_E^S$.

Let us consider now the quasi-order structure $(\Psi_E^S, <)$. Like L_E^x, also L_E^S with the induced relation $<$ [hence, more specifically, $(\Psi_E^S, <)$] can be considered an empirical logic constructed by applying the general procedures described in Section 2.5. Indeed, it can be obtained by selecting a subset of epistemically accessible wffs in L that can be obtained from the (metalinguistic) scheme $(\forall \mathbf{x})(\mathbf{S}(\mathbf{x}) \to \mathbf{E}(\mathbf{x}))$, by designating these wffs with new symbols, taking them as atomic wffs of L_E^S, and by introducing new symbols of operation on this subset whose semantic properties follow from semantic relations between formulas in the subset. Thus, $(\Psi_E^S, <)$, like $(\Psi_E^x, <)$, can be treated from two different viewpoints, the first one arising from the definition of $<$ and from statement (i) in Proposition 3.2.1, the second one arising from statement (ii) in Proposition 3.2.1. According to the former, the lattice structure of $(\Psi_E^S/\approx, <)$ is isomorphic to the lattice structure of $(\mathscr{E}_E, <)$ and induced by it, so that it simply reflects, via our intended physical interpretation, the empirical structure of the set of physical properties. According to the latter, the quasi-order relation $<$ on Ψ_E^S coincides with the quasi-order induced on Ψ_E^S by the family of truth functions $\{f_i^S\}_{i \in I, S \in \mathscr{S}}$ (where f_i^S, which does not depend on i if the FI condition holds, is completely determined by the restriction of $f_{i\sigma}$, with any $\sigma \in \Sigma$, to the atomic wffs of L, hence by the contingent of $f_{i\sigma}$); in this sense, $(\Psi_E^S, <)$ can be considered a logical structure and $(\Psi_E^S/\approx, <)$ its algebra of propositions [of course, $(\Psi_E^S, <)$ maintains an empirical character even if this perspective is embraced].

Whenever the second viewpoint is adopted, we have again a structure [to be precise, $(\Psi_E^S/\approx, <)$] that occurs both in CP and in QP and exhibits the main basic mathematical features usually assumed when an algebraic semantics for QL is assigned, the properties lacking being recovered through $(\mathscr{E}_E, <)$ whenever CP or QP are considered; moreover, the interpretation of Ψ_E^S coincides with a possible interpretation in QL [see in particular Beltrametti and Cassinelli (1976, 1981)]. Thus, we can claim that we have derived another quantum logical structure in our extended classical framework, on the basis of physical assumptions [the syntactic differences between $(\Psi_E^x, <)$ and $(\Psi_E^S, <)$ being the counterpart of the differences in the interpretations]. Again, this derivation makes classifying $(\Psi_E^S, <)$ as a new logic philosophically questionable.

As in the case of L_E^x, it is important to notice that the connectives $^\perp$, $\sqcap$, and $\sqcup$ in the alphabet of L_E^S cannot be *a priori* identified with the connectives $\neg$, $\wedge$, and $\vee$, respectively, in the alphabet of L; indeed, their interpretation must be deduced from the above interpretation of the wffs in Ψ_E^S [for instance, for every $\mathbf{E}_1, \mathbf{E}_2 \in \mathscr{E}_E$, $\subset(\mathbf{S}, \mathbf{E}_1) \sqcup \subset(\mathbf{S}, \mathbf{E}_2)$ states that all physical objects prepared according to the state denoted by $\mathbf{S}$ have the physical property denoted by $\mathbf{E}_1 \sqcup \mathbf{E}_2$]. The identification of $\neg$ and $^\perp$ is not possible here; indeed, they could be seen as examples of van Fraassen's (1974) exclusion and choice negations, respectively. We also stress that any truth function f_i^S is two-valued, as we expect if it is interpreted as attributing a truth value to the statements of Ψ_E^S.

Let us consider now the family of fuzzy translation mappings. Here, the mapping ϑ_r^S from $(\vartheta_r^S)_{r\in[0,1]}$ maps any **A**(**S**) that belongs to Ψ_E^S on the wff $(\pi_r \mathbf{x})\mathbf{E}^A(\mathbf{x})/\mathbf{S}(\mathbf{x})$; the latter belongs to Ψ and can be interpreted, according to the rules in Section 1.5, as follows:

"The physical objects prepared according to the state denoted by **S** have the physical property denoted by $\mathbf{E}^A$ with frequency r."

This suggests that the following new interpretation of **A**(**S**) can be introduced:

"A physical object prepared according to the state denoted by **S** has the physical property denoted by $\mathbf{E}^A$."

Whenever this interpretation is adopted the number $r_i = \varphi_i^S(\mathbf{A}(\mathbf{S}))$ (which does not depend on i if the FI condition holds) can be interpreted as the "degree of truth" of **A**(**S**).

In order to understand the genesis of this "fuzzy logical" viewpoint (which underlay our choice of the name "fuzzy-truth function" for φ_i^S), let us observe that Definition 3.2.2, Proposition 1.6.2(i), and Definition 1.4.1(v), imply, whenever the measure function ν_i reduces to the number of elements of its argument, that $\varphi_i^S(\mathbf{A}(\mathbf{S}))$ is the percentage of elements in $\rho_i(\mathbf{S})$ that belong to $\rho_i(\mathbf{E}^A)$; hence, it can directly be interpreted as a physical frequency, consistent with our arguments in Section 1.5. Alternatively, it can be interpreted as the "degree of truth (in the laboratory i) of the property denoted by $\mathbf{E}^A$ for a physical object prepared according to the state denoted by **S**," consistent with the standard interpretation of many-valued truth functions in fuzzy logics (however, we think that this interpretation is problematic, since it implies the adoption of a non-Tarskian theory of truth, and introduce it here only because of its occurrence in the literature).

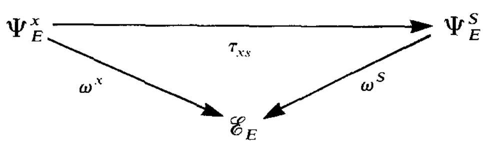
Let us come now to the quasi-order structure $(\Psi_E^S, <)$. Here, again, $(\Psi_E^S, <)$ can be treated from two different viewpoints because of statements (i) and (ii) in Proposition 3.2.1; it is apparent that all our arguments above regarding our first interpretation of L_E^S can be repeated with the substitution of the family $\{\varphi_i^S\}_{i\in I, S\in\mathscr{S}}$ to $\{f_i^S\}_{i\in I, S\in\mathscr{S}}$. Therefore, we conclude that we have a structure which occurs both in CP and QP and exhibits the basic mathematical features usually assumed when an algebraic semantics for QL is assigned [again, the properties lacking can be recovered through $(\mathscr{E}_E, <)$ whenever CP or QP are considered]; moreover, Ψ_E^S has an interpretation which is a possible interpretation in many-valued QL [see, in particular, Watanabe (1969)]. Thus, we can say that we have derived an infinite-valued quantum logical structure in our extended classical framework, on the basis of physical assumptions, again noticing that considering these structures as true "logics" is philosophically problematic (it should be noted that our procedures exemplify a general method for deriving and interpreting multi-valued logical structures in a classical framework; in particular, they could be applied in order to obtain three-valued quantum logics].

Even in this case, we remark that the connectives $^\perp$, $\sqcap$, and $\sqcup$ in L_E^S cannot be identified *a priori* with the connectives $\neg$, $\wedge$, and $\vee$ in L; indeed, their interpretation must be deduced from the above interpretation of Ψ_E^S [for instance, for every $\mathbf{E}_1, \mathbf{E}_2 \in \mathscr{E}_E$, $\subset(\mathbf{S}, \mathbf{E}_1) \sqcup \subset(\mathbf{S}, \mathbf{E}_2)$ states that a physical object prepared according to the state denoted by $\mathbf{S}$ has the physical property denoted by $\mathbf{E}_1 \sqcup \mathbf{E}_2$].

Finally, let us briefly explore the links between L_E^x and L_E^S. We have already seen that the quotient structures $(\Psi_E^x/\approx, <)$ and $(\Psi_E^S/\approx, <)$ are both isomorphic to $(\mathscr{E}_E, <)$. In addition, let us observe that the mapping τ_{xs}, recursively defined by

for every $\mathbf{E} \in \mathscr{E}_E$,	$\tau_{xs}(\mathbf{E}(\mathbf{x})) = \subset(\mathbf{S}, \mathbf{E})$
for every $\mathbf{A}(\mathbf{x}) \in \Psi_E^x$,	$\tau_{xs}(\mathbf{A}^\perp(\mathbf{x})) = (\tau_{xs}(\mathbf{A}(\mathbf{x})))^\perp$
for every $\mathbf{A}(\mathbf{x}), \mathbf{B}(\mathbf{x}) \in \Psi_E^x$	$\tau_{xs}(\mathbf{A}(\mathbf{x}) \sqcap \mathbf{B}(\mathbf{x})) = \tau_{xs}(\mathbf{A}(\mathbf{x})) \sqcap \tau_{xs}(\mathbf{B}(\mathbf{x}))$
	$\tau_{xs}(\mathbf{A}(\mathbf{x}) \sqcup \mathbf{B}(\mathbf{x})) = \tau_{xs}(\mathbf{A}(\mathbf{x})) \sqcup \tau_{xs}(\mathbf{B}(\mathbf{x}))$

is a quasi-order isomorphism of $(\Psi_E^x, <)$ onto $(\Psi_E^S, <)$, which makes the diagram



commutative. This clarifies the deep reasons for the structural analogies between Ψ_E^x and Ψ_E^S.

Looking at the above diagram, one can suspect that the isomorphism between Ψ_E^x and Ψ_E^S can be extended to the semantic level (in the sense that corresponding formulas have the same truth values). Now, let $i \in I$ and let the interpretation of the variables $\sigma \in \Sigma$ be such that the condition $\rho_i^\sigma(\mathbf{x}) \in \rho_i^\sigma(\mathbf{S})$[equivalently, $f_{i\sigma}(\mathbf{S}(\mathbf{x})) = 1$] holds.

Then, by making use of the equivalence

$$f_i^S(\mathbf{A}(\mathbf{S})) = 1 \quad \text{iff} \quad f_{i\sigma}((\forall \mathbf{x})(\mathbf{S}(\mathbf{x}) \to \mathbf{E}^A(\mathbf{x}))) = 1$$

which follows from Definition 3.2.2, we get

$$f_i^S(\mathbf{A}(\mathbf{S})) = 1 \quad \text{implies} \quad f_{i\sigma}(\mathbf{E}^A(\mathbf{x})) = 1$$

Since

$$\begin{aligned} f_{i\sigma}^x(\tau_{xs}^{-1}(\mathbf{A}(\mathbf{S}))) &= f_{i\sigma}(\omega^x(\tau_{xs}^{-1}(\mathbf{A}(\mathbf{S})))(\mathbf{x})) \\ &= f_{i\sigma}(\mathbf{E}^A(\mathbf{x})) \end{aligned}$$

we obtain

$$f_i^S(\mathbf{A}(\mathbf{S}))=1 \quad \text{implies} \quad f_{i\sigma}^x(\tau_{xs}^{-1}(\mathbf{A}(\mathbf{S})))=1$$

It is easy to see that the above implication cannot be reversed unless

$$f_{i\sigma}(\mathbf{S}(\mathbf{x})\to\mathbf{E}^A(\mathbf{x}))=1 \quad \text{implies} \quad f_{i\sigma}((\forall\mathbf{x})(\mathbf{S}(\mathbf{x})\to\mathbf{E}^A(\mathbf{x})))=1$$

This implication holds in CP whenever **S** denotes a pure state (see below, Section 3.4), but does not generally hold in QP. Thus, while the truth of a wff $\mathbf{A}(\mathbf{S})\in L_E^S$ implies (under suitable conditions) the truth of the wff $\mathbf{A}(\mathbf{x})\in L_E^x$ such that $\tau_{xs}(\mathbf{A}(\mathbf{x}))=\mathbf{A}(\mathbf{S})$, the converse statement is generally false, and it may be erroneous to transfer truth values from wffs in Ψ_E^x to the corresponding (via τ_{xs}) wffs in Ψ_E^S. In my opinion, such an abusive transfer plays an important role in some classical "paradoxes" in QP.

Bearing in mind the above interpretations of L_E^x and L_E^S, and restricting consideration, for the sake of simplicity, to two-valued semantics, one can point out another epistemologically relevant difference between these languages which obtains whenever the FI condition holds. More precisely, recall that we have repeatedly affirmed that L_E^x and L_E^S are obtained by picking out suitable subsets of epistemically accessible wffs in L (see Section 2.5 and the first part of the present subsection). Now, we observe that testing the truth value of a wff $\mathbf{A}(\mathbf{x})\in\Psi_E^x$ in a laboratory $i\in I$ under the interpretation $\sigma\in\Sigma$ implies: (i) considering the physical object $\rho_i^\sigma(\mathbf{x})$, which necessarily belongs to a set $\rho_i(\mathbf{S})$, with **S** a symbol of state; (ii) performing a measurement on $\rho_i^\sigma(\mathbf{x})$ by means of an apparatus which actualizes in i the exact effect denoted by $\mathbf{E}_A$. After the measurement, which is equivalent to applying a new preparation procedure in the laboratory, the physical object loses its previous identity and transforms into a new physical object which does not necessarily belong to $\rho_i(\mathbf{S})$ (a similar transformation occurs whenever dynamical evolution is considered; yet we know the state of the physical object at any time in this case).

Thus, the physical object could not be disposable for further measurements: each formula in L_E^x is epistemically accessible, but different formulas could be not simultaneously accessible.

Let us come to L_E^S. Testing the truth value of a wff $\mathbf{A}(\mathbf{S})\in\Psi_E^S$ is equivalent to testing the truth value of the wff $(\forall\mathbf{x})(\mathbf{S}(\mathbf{x})\to\mathbf{E}^A(\mathbf{x}))\in\Psi$. Hence, it implies: (i) considering all the physical objects that belong to $\rho_i(S)$; (ii) performing a measurement on each of these objects by means of an apparatus which actualizes in i the exact effect denoted by $\mathbf{E}^A$. Again, after the measurements, the physical objects that were in $\rho_i(\mathbf{S})$ could be no longer disposable for further measurements. Yet, the truth value of another formula in Ψ_E^S can be tested in a different laboratory j, with the same methods described above; whenever the FI condition holds, statement (iii) in Proposition 3.2.1 guarantees that this test gives the same result as a test performed in *i*. Thus, the formulas of L_E^S, which are epistemically accessible, are also simultaneously accessible. This result is epistemologically important and throws further light on the different roles that can be attributed to L_E^x and L_E^S in physics.

3.4. Classical versus Quantum Physics

We have seen in Section 2 that the preclusivity relation $\#$ on the set of states plays a fundamental role in our approach; in particular, the lattice structure of $\mathscr{L}$ and the definitions of fuzzy and exact effects rest on the definition of $\#$. Thus, it is important that the difference between a classical and a quantum semantic model can be established as a difference between preclusivity relations.

Let us consider CP. By using common physical language, we can say that the following assumption is fundamental (though usually implicit) in classical theories: the minimal effect associated with a state gives the no answer whenever applied to any physical object in another state if the latter state is decomposable (in the sense established by classical statistical mechanics) by means of pure states which do not appear in the decomposition of the former (hence, for every state the decomposition by means of pure states is unique; furthermore, the minimal effect associated with a pure state gives the no answer if it is applied to any physical object in another pure state). By translating this statement, via our intended interpretation, into our present framework, we say that a physical model $\mathfrak{M}$ is a *classical physical model* whenever it satisfies the EO, FI, and MS conditions (see Definition 2.6.1) and, in addition, the following *classical physics condition* (briefly, CP condition) holds.

CP. Let $\mathbf{S}_1, \mathbf{S}_2 \in \mathscr{S}$, and let $(\mathbf{S}_{1k})_{k\in K}$ and $(\mathbf{S}_{2j})_{j\in K}$ be families of mutually preclusive symbols of pure states associated to $\mathbf{S}_1$ and $\mathbf{S}_2$, respectively, which satisfy statement (i) in the MS condition.

Then, $\mathbf{S}_1 \# \mathbf{S}_2$ iff for every $k, j \in K$, $\mathbf{S}_{1k} \neq \mathbf{S}_{2j}$.

Whenever $\mathfrak{M}$ is a classical physical model, the following statements hold (Garola, 1989).

(i) Let $\mathbf{S}_1, \mathbf{S}_2 \in \mathscr{S}_P$; then, $\mathbf{S}_1 \# \mathbf{S}_2$ iff $\mathbf{S}_1 \neq \mathbf{S}_2$.

(ii) For every $\mathbf{S} \in \mathscr{S}$, a unique family $(\mathbf{S}_k)_{k\in K}$ exists which satisfies statement (i) in the MS condition.

(iii) The weak orthocomplementation $^\perp$ on $\mathscr{P}(\mathscr{S}_P)$ reduces to the set-theoretic complementation ^{C}, that is, for every $H \in \mathscr{P}(\mathscr{S}_P)$, $H^\perp = H^C = \mathscr{S}_P \backslash H$ [hence, the set $\mathscr{L} = \{H \in \mathscr{P}(\mathscr{S}_P) \mid H = H^{\perp\perp}\}$ coincides with $\mathscr{P}(\mathscr{S}_P)$].

(iv) The set $\mathscr{E}_O$ coincides with the set $\mathscr{E}$.

(v) The restriction S_{TE} of S_T to $\mathscr{E}_E$ is an order isomorphism of the complete orthocomplemented lattice $(\mathscr{E}_E, \mathbb{O}, <, ^\perp)$ [equivalently,

$(\mathscr{E}_E, \mathbb{0}, <, ')]$ onto $(\mathscr{P}(\mathscr{S}_P), \varnothing, \subseteq, {}^C)$, which preserves the orthocomplementation [hence, $(\mathscr{E}_E, \mathbb{0}, <, {}^\perp)$ is distributive, and for every $E \in \mathscr{E}_E$ the certainly false domain $S_F(\mathbf{E})$ coincides with $\mathscr{S}_P \backslash \mathbf{S}_T(\mathbf{E})$].

(vi) For every $\mathbf{E} \in \mathscr{E}_E$ and $i \in \tilde{I}$,

$$\rho_i(\mathbf{E}) = \bigcup_{S \in S_T(E)} \rho_i(g(\mathbf{S}))$$

(vii) For every $\mathbf{E}_1, \mathbf{E}_2 \in \mathscr{E}_E$ and $i \in \tilde{I}$,

$$\rho_i(\mathbf{E}_1 \cap \mathbf{E}_2) = \rho_i(\mathbf{E}_1) \cap \rho_i(\mathbf{E}_2)$$

$$\rho_i(\mathbf{E}_1 \cup \mathbf{E}_2) = \rho_i(\mathbf{E}_1) \cup \rho_i(\mathbf{E}_2)$$

[in addition, for every $\mathbf{E} \in \mathscr{E}_E$, $\rho_i(\mathbf{E}^\perp) = D_i \backslash \rho_i(\mathbf{E})$, which holds independently of the CP condition].

(viii) For every $i \in \tilde{I}$, $\sigma \in \Sigma$, $\mathbf{x} \in X$, and $\mathbf{A}(\mathbf{x})$, $\mathbf{B}(\mathbf{x}) \in \Psi_E^x$,

$$f_{i\sigma}^x(\mathbf{A}(\mathbf{x}) \cap \mathbf{B}(\mathbf{x})) = f_{i\sigma}(\tau^x(\mathbf{A}(\mathbf{x})) \wedge \tau^x(\mathbf{B}(\mathbf{x})))$$

$$f_{i\sigma}^x(\mathbf{A}(\mathbf{x}) \cup \mathbf{B}(\mathbf{x})) = f_{i\sigma}(\tau^x(\mathbf{A}(\mathbf{x})) \vee \tau^x(\mathbf{B}(\mathbf{x})))$$

[in addition, for every $\mathbf{A}(\mathbf{x}) \in \Psi_E^x$, $f_{i\sigma}^x(\mathbf{A}^\perp(\mathbf{x})) = f_{i\sigma}(\neg\tau^x(\mathbf{A}(\mathbf{x})))$, which holds independently of the CP condition].

(ix) Let $i \in \tilde{I}$, $\mathbf{S} \in \mathscr{S}_P$. Then, for every $\mathbf{A}(\mathbf{S}) \in \Psi_E^S$, $\varphi_i^S(\mathbf{A}(\mathbf{S})) = f_i^S(\mathbf{A}(\mathbf{S}))$ [hence, $\varphi_i^S(\mathbf{A}(\mathbf{S})) \in \{0, 1\}$]. Furthermore, let $\sigma \in \Sigma$, let $\mathbf{x} \in X$ be such that $\rho_i^\sigma(\mathbf{x}) \in \rho_i$ (S), and let τ_{xs} be the mapping defined at the end of Section 3.3. Then, for every $\mathbf{A}(\mathbf{x}) \in \Psi_E^x$, $f_{i\sigma}^x(\mathbf{A}(\mathbf{x})) = f_i^S(\tau_{xs}(\mathbf{A}(\mathbf{x})))$.

Let us come now to QP. Here, the CP conditions is untenable; indeed, pairs of different pure states exist in QP such that the minimal effect associated with one of them may give the yes answer if applied to a physical object prepared according to the other (see, in particular, Section 2.2). By formalizing this result in our present framework, we obtain that, for every $\mathbf{S}_1$, $\mathbf{S}_2 \in \mathscr{S}_P$, $\mathbf{S}_1 \neq \mathbf{S}_2$ does not imply $\mathbf{S}_1 \# \mathbf{S}_2$, which contradicts our above statement (i).

In place of the CP principle, some physical assumptions are made in QP which make $(\mathscr{E}_E, \mathbb{0}, <, ')$ a weakly modular lattice satisfying the covering law; more directly, one could simply assume these properties in the complete orthocomplemented (atomic) lattice $(\mathscr{E}_E, \mathbb{0}, <, ')$, resting on well-known treatments of the foundations of QP for a physical justification of these assumptions [see, in particular, Piron (1976)].

In any case, one may wonder whether one or more of statements (i)–(ix) still hold in QP. Now it can be proved that, whenever our interpretation is maintained, all these statements have counterexamples in QP [with the possible exception of statement (iv)].

In particular, the lattice $(\mathscr{E}_E, \mathbb{0}, <, ')$ is not distributive [at least for irreducible, or not completely reducible, quantum systems (e.g., Jauch, 1968;

Piron, 1976)] and for every $\mathbf{E} \in \mathscr{E}_E$ there are some pure states which neither belongs to $\mathbf{S}_T(\mathbf{E})$ nor to $\mathbf{S}_F(\mathbf{E})$; these statements obviously contrast with statement (v). Furthermore, the fuzzy-truth function φ_i^S on Ψ_E^S is not two-valued, not even when $\mathbf{S} \in \mathscr{S}_P$, contrary to statement (ix). More important, the connectives $\sqcap$ and $\sqcup$ in $\mathscr{A}_E^x$ cannot be identified with the logical connectives $\wedge$ and $\vee$ in $\mathscr{A}$ [contrary to statement (viii)] and the language L_E^x is not identifiable with L_E^S [contrary to statement (ix)] even if it can be formally translated onto L_E^S by means of the bijective mapping τ_{xs} introduced in Section 3.3; indeed, the truth functions $f_{i\sigma}^x$ and f_i^S defined on L_E^x and L_E^S do not necessarily take the same values on corresponding formulas, so that these languages are not semantically isomorphic (in particular, the truth or falsity of some statements on a physical object prepared according to a given state does not authorize one to infer the truth or falsity of the corresponding statements about all the objects prepared according to that state).

We note that statements (i)–(ix), when interpreted according to our intended interpretation, formally express some deeply impressed and intuitive beliefs which have been sometimes assumed (often implicitly, especially by those physicists who share a "realistic" attitude) as epistemological requirements for every physical theory. Thus, their failure in QP has been considered paradoxical by some authors. Partly, this attitude depends on the (erroneous) conviction that the necessity of abandoning some epistemological assumptions implies the need to adopt a new kind of logic.

In my opinion, it is an advantage of the present approach that the aforesaid requirements appear as a consequence of an explicit "*a priori*" condition, the CP condition, that has no "logical" necessity (as a matter of fact, it turns out to be false in microphysics), while the quantum physical features discussed above lose their paradoxical aspects in our perspective and appear as a consequence of the rejection of the same condition. Of course, the epistemological aspects of this abdication may cause trouble and be the origin of debate (in particular, the breakdown of strict determinism; see Section 2.2; more specifically, the assumption, which is anomalous from a classical viewpoint, that a physical object prepared according to some pure state may be registered by a device testing whether the object is prepared according to another state).

In any case, our approach shows that no nonclassical logical system is strictly needed when founding QP. Thus, in conclusion, QP does not require an alternative logic (though it can suggest new logical structures, as we have seen above), but, rather, a nonclassical epistemological attitude.

I close with two general remarks. First, observe that our neat distinction between different quantum logical languages, founded on a unique extended classical language, besides being useful in the analysis of some "paradoxes" in QP, also allows for a reformulation of some classical problems, such as the problem of completeness of quantum laws (Garola 1989), which is strictly connected with the endless debate about the interpretation of quantum mechanics (e.g., Blokhintsev, 1968).

Second, note that, when adopting known procedures in QL [e.g., Putnam (1969) and Finkelstein (1969, 1972); according to Holdsworth and Hooker (1983), these procedures "read off" the logic from the lattice of subspaces] to our framework, the Hilbert space model for QP can be adopted in order to introduce a semantic model $\mathfrak{M}^s$ for the language L_E^S, with a two-valued truth function $\bar{f}^S$; this can be done by means of standard correspondences, and it can be easily proved that $\bar{f}^S$ coincides with the truth function f^S in this case. Thus, the language L_E^S with truth function f^S can be obtained by substitution of Hilbert spaces to set theory in the semantic model, without any reference to L. However, it must be clearly understood that this procedure implies the adoption of a non-Tarskian truth theory, which is a higher price to pay in my opinion.

It is also interesting to remark that no similar procedure exists for the individual language L_E^x, because of the absence of any element in the Hilbert space model which is associated with individual physical systems.

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